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RICCATI TRANSFORMATIONS AND OSCILLATION OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS WITH DAMPING TERM

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سورة يونس، الآية [10]

Declaration

I am Maryam Mohammed Abdalsalam Hamed confirm that the work in this thesis, unless otherwise referenced is researcher`s own work, and has not been previously submitted to meet requirements of an award at this University or any other education or research institution, I furthermore, cede copyright of this thesis in favour of Sirt University.

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Dedication

To the one who taught me that life is a journey of struggle, and that knowledge is its most powerful weapon...

To my first teacher, whose words have always guided me like stars lighting my path...

To the one who witnessed the beginning of this journey and patiently awaited its end...

To the one who instilled in me the value of learning and the meaning of perseverance...

I dedicate the fruit of this effort to his pure soul, hoping it will be a light added to the scale of his good deeds.

(My father – may Allah have mercy on him)

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Finally, acknowledgment is given to all those who contributed, directly or indirectly, to the successful completion of this thesis.

It is hoped that this work will serve as a useful contribution to the field and support future studies.

Abstract

Difference equations are the discrete counterpart to ordinary differential equations. Instead of describing continuous rates of change, these equations describe changes between function values at discrete points. This oscillatory behavior allows the concepts of calculus and analysis to be extended to discrete environments, which is crucial in fields such as computer science and digital signal processing where data are inherently discrete. This study highlights the oscillatory behavior of solutions of some second-order nonlinear difference equations with a damping coefficient, given their great importance in applied mathematics and dynamic analysis.

Using methods like Ricatti type transformations a number of oscillation conditions were obtained for a certain class of difference equations, consequently some previous theories were expanded. Examples that demonstrated the results were be to support the hypotheses put forward in this study.

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List of Symbols

Symbol	Description
Δ	The forward difference operator
I	The identity operator
E	The shift operator
$\mathbb{N} = \mathbb{N} \cup \{0\}$	The set of natural numbers including zero
$\mathbb{R}$	The set of real numbers
$\prod$	Product symbol
$\sum$	Summation symbol

Introduction

In biology, ecology, physiology, physics, engineering and economics, difference equations appear as naturally as discrete analogues and as numerical solution to differential and delay differential equations. Determining the oscillation requirements for difference equations has drawn a lot of interest in the literature lately. For example, a great deal of work has been done on this subject by (Agarwal et al., 1996), (Agarwal et al., 1997), (Arul, 1998), (Cheng & Li, 1989), (Cheng & zhang, 1994), (Cheng & Lin, 1998), (Cheng & Patula, 1993), (Graef & Spikes, 1994), (Greaf et al. 1996), (Leads & Gyori 1991), (Hooker & Patula 1983), (Lalli et al., 1992), (Manuel,1997), (Pandian,1995), (Popenda,1995), (Szmanda,1997), (Stavroulakis, 1989),(Stavroulakis, 1994), (Stavroulakis, 1995) , (Zafar & Dahiya, 1993).

(Agarwal,1992) and (Agarwal & Wong, 1997), (Elaydi, 1996), (Freedman, 1980), (Hale & Kocak, 1991), (Kelly & Peterson, 1991), (Lakshmikantham & Trigiante, 1988), (Potts, 1981), (Suzuki, 1996), and (Weil, 1980).

A difference equation in one independent variable n $\in \mathbb{N}$ and one unknown x_n is a functional equation of the form $f(n, x_n, \dots, x_{n+m}) = 0$ where f is a given function of n and the values x_n at n $\in \mathbb{N}$. If we defined the forward difference operator Δ as $\Delta x_{n=}x_{n+1} - x_n$ and $\Delta^m x_n = \Delta(\Delta^{m-1} x_n)$, then using the operator Δ the above equation can be written in the form

$$g(n, x_n, \Delta x_{n+1}, \dots, \Delta^m x_n) = 0. \tag{1.1}$$

It was this notation which led (1.1) to the name, difference equation.

Even though nonlinear difference equations are more difficult to analyze than linear equations since some linear representations do not yield a sufficiently accurate model for physical systems. Furthermore, not all nonlinear systems can be linearized and still maintain the integrity of modeled phenomena. Thus, the study of nonlinear equations is both an interesting and useful area of research.

It is widely acknowledged that most nonlinear difference equations of order greater than one cannot be solved explicitly. Nevertheless, solving these equations is essential to gaining a deeper understanding the systems of which they govern. With the use of a computer, a great number of numerical solutions to these equations can be obtained. As a result, continuous problems are often approximated by discrete ones.

However, any computational result must also be proven analytically. Therefore, having closed-form analytical solutions, or at least insights into the qualitative behavior of solutions, is of great importance.

In the qualitative analysis of difference equations, the oscillatory behavior of solutions plays a pivotal role. This behavior is evident in various mechanical processes, when there is a correspondence or analogy between the outcomes of oscillatory theory in the discrete and continuous settings, applying results from one domain to the other is not straightforward it requires specific approaches or techniques. Hence, investigating the oscillatory nature of solutions to difference equations is a subject of considerable significance.

Outlines of The Thesis

The oscillation theory of second-order a nonlinear damping term difference equations and deriving oscillation conditions for their solutions is the subject of this thesis. In the proofs, we employ the Riccati transform to obtain new conditions that are much different or more general than the ones imposed in the literature. Further, examples are cited wherever necessary to illustrate the main results.

The plan of the thesis is as follows:

Chapter One:

An introduction to difference equations is provided in this chapter. It includes a summary of the key terms and symbols for difference equations that will be used in the upcoming chapters, which are helpful for extending the findings of this thesis. It includes a few examples to illustrate the significance of the difference equations. The oscillation problem of second order differential equations is the subject of this introductory chapter.

Chapter Two:

Reviews of the literature are given in this chapter along with some background information.

Chapter Three:

Chapter three, is devoted to do analytical comparison between the results of (Thandapani & Pandian, 1995) and (Thandapani, 1995).

Chapter Four:

The adequate conditions for the oscillation of the solutions of the second order non-linear difference equation of the form:

$$\Delta(c_m \Delta x_m) + \theta_m \, \varphi(x_m, \Delta x_m) + q_{m+1} f(x_{m+1}) = 0, m \in \mathbb{N} \qquad (E)$$

will be examined in this chapter. Examples will be provided to demonstrate the newly discovered results of the second order nonlinear difference equations.

Chapter Five:

This chapter summarizes the findings and makes recommendations for additional research.

Chapter One

Introductory Concepts

1.1 Introduction

A difference equation is a specific type of recurrence relation where each term in a sequence is determined by a mathematical rule that references earlier terms. In recent years, there has been an increasing focus on the study of difference equations in research. A range of approaches and strategies have been employed to investigate the qualitative characteristics of their solutions. Second-order nonlinear difference equations have attracted significant scholarly attention due to their numerous applications. The existence and uniqueness of solutions, their stability, and their oscillatory behavior are important topics of study. However, further research is necessary to ascertain whether solutions to second-order difference equations of integer order demonstrate oscillatory behavior. In a multitude of real-world applications, including control systems, electrodynamics, liquid mixing, neutron transport, networks, and population dynamics, equations containing damping terms are of particular significance. The examination of the oscillations of the solutions is imperative when conducting a qualitative analysis of these systems, particularly when their growth is contingent on both past and present states, as well as past rates of change. Consequently, a substantial corpus of literature has emerged concerning the oscillatory behavior of second-order difference equations with damping terms.

The transformation of second-order difference equations into linear forms is a pivotal step in the field of mathematical analysis, a process that is facilitated by Riccati transforms. This transformation enables a more profound examination of their dynamic properties. The software provides a systematic method for deriving solutions to nonlinear equations, thereby illuminating the fundamental rhythmic behaviors of dynamic systems. As discussed in academic literature, the Riccati transformation facilitates the identification of solution variations that bear similarity to those explored in fractional Kairat equations. This underscores the system's adaptability in the context of nonlinear systems research across diverse disciplines, including physics and engineering. Additionally, the Riccati framework makes it easier to assess important solution properties like stability and periodicity, which are crucial for ferroelectric material modeling. Research on these materials has been demonstrated to advance knowledge and advance applications like capacitors and non-volatile memory. This highlights how important Riccati transformations are to mathematical analysis.

Will provide a variety of examples to clarify the importance of difference equations.

Example 1.1 (Kelley & Peterson, 2001)
Consider the solution of the differential equation.

$$\ddot{y}(t) = ty(t)$$

When modelling physical systems that undergo abrupt changes, such as quantum mechanical tunneling effects and irregular wave patterns, the Airy equation is helpful. It also contributes to the description of particle motion in quantum physics as determined by the Schrödinger equation.

Investigating power series solutions of the form is one strategy.

$$y(t) = \sum_{m=0}^{\infty} b_m t^m$$

Substitution of the series into the differential equation results in

$$\sum_{m=2}^{\infty} m(m-1)b_m t^{m-2} = \sum_{m=0}^{\infty} b_m t^{m+1}$$

The shift of index $m \to m+3$ in the series on the left side of the previous gives us

$$\sum_{m=-1}^{\infty} b_{m+3}(m+3)(m+2)t^{m+1} = \sum_{m=0}^{\infty} b_m t^{m+1}$$

For these series to be equivalent over a range of *t* values, the coefficients of t^{m+1} need to match for every $m = -1, 0, \ldots$

when $m = -1$, we have

$$b_2 \times 2 \times 1 = 0,$$

so $b_2 = 0$.

When $m \in \mathbb{N}$ $\quad b_{m+3}(m+3)(m+2) = b_m$, or

$$b_{m+3} = \frac{b_m}{(m+3)(m+2)} \tag{1.1}$$

The equation (1.1) is a difference equation that, theoretically, enables us to use the initial coefficients b_0 and b_1 to calculate all subsequent coefficients b_m.

Note that

$$b_{3m+2} = 0 \text{ for } m = 0,1, \ldots$$

Given that $b_2 = 0$ and treating b_0 and b_1 as arbitrary constants, we arrive at the general solution to the Airy equation, expressed as a power series:

$$y(t) = b_0\left(1 + \frac{t^3}{3.2} + \frac{t^6}{6.5.3.2} + \cdots\right) + b_1\left(t + \frac{t^4}{4.3} + \frac{t^7}{7.6.5.3} + \cdots\right)$$

Returning to the difference equation, we consider the following:

$$\frac{b_{m+3}}{b_m} = \frac{1}{(m+3)(m+2)} \to 0 \text{ as } m \to \infty$$

The ratio test indicates that the power series converges for every value of the variable t.

Example 1.2 (Kelley & Peterson, 2001)
In 1626, it is said that Peter Minuit acquired Manhattan Island for goods valued at approximately $24. If that $24 had been invested in 1626 with a 7% annual interest rate, compounded quarterly, what would its worth be by 1986?
Let $y(t)$ denote the value of investment after t quarters of a year.

Then

$$y(0) = 24.$$

The following difference equation is satisfied by $y(t)$ since the interest rate is 1.75% per quarter.

$$y(t+1) = y(t) + 0.0175\, y(t)$$

$$= (1.0175)y(t), \quad t \in \mathbb{N}$$

Computing y recursively, we have

$$y(1) = 24(1.0175),$$

$$y(2) = 24(1.0175)^2,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$y(t) = 24(1.0175)^t.$$

After a period of 360 years, equivalent to 1,440 quarters, the investment's value can be determined as follows

$$y(1440) = 24(1.0175)^{1440}$$

$$\approx 1.697 \times 10^{12}.$$

Example 1.3 (Kelley & Peterson, 2001)

Euler's approach provides a foundational numerical scheme applied to approximate the solution of initial value issues linked with ordinary differential equations, by discretizing the domain and repeatedly producing successive approximations

$$\acute{y}(t) = f\big(t, y(t)\big)$$

$$y(t_0) = t_0,$$

this method is derived by approximating the derivative $\acute{y}(t)$ using the forward difference quotient $\frac{y(t+h)-y(t)}{h}$, where h denotes a sufficiently small step size. Accordingly, we obtain the recursive relation:

$$\frac{y(t+h) - y(t)}{h} = f\big(t, y(t)\big)$$

$$y(t+h) = y(t) + hf\big(t, y(t)\big)$$

To express the difference equation in a more standard discrete form, we define the sequence $y_n = y(t_0 + nh)$ for $n = 0,1,2, \dots$ Then Under this substitution, the continuous time problem is reformulated in terms of a discrete sequence as follow

$$y_{n+1} = y_n + h\, f(t_0 + n\, h, y_n)$$

Given that y_0 is specified by the initial condition, the approximated values y_n can be generated recursively. However, the validity and accuracy of these approximations are typically confined to a limited range of n, depending on the behavior of the function $f(t, y)$ and the step size h. For instance, consider the case where $f(t, y) = y^2$, and $t_0 = 0$. The corresponding initial value problem then admits the analytical solution

$$y(t) = \frac{y_0}{1 - y_0 t}$$

This analytical solution exhibits a singularity, or "blows up," $t = \frac{1}{y_0}, y_0 \neq 0$, indicating that the solution becomes unbounded in finite time. In contrast, the solution to the corresponding nonlinear difference equation, given by

$$y_{n+1} = y_n + h y_n^2$$

y_0 specified, remains defined for all n. This represents a fundamental example of a nonlinear difference equation, highlighting the divergence in behavior between continuous and discrete models.

This thesis will explore the necessary conditions for the oscillation of solutions to the second-order non-linear difference equation structured as follows:

$$\Delta(c_m \Delta x_m) + \theta_m\, \varphi(x_m, \Delta x_m) + q_{m+1} f(x_{m+1}) = 0, m \in \mathbb{N} \qquad (E)$$

1.2 Basic Concepts

Definition 1.2.1 (Elaydi 2005)

A difference equation is an equation that represents the changes in a variable. Its classification is based on several factors, including its order, whether it is autonomous or non-autonomous, and whether it is linear or nonlinear.

Definition 1.2.2 (Schwarz, 2007)

The order of the difference equation is determined by the difference between the highest and the lowest the indices presented in the difference equation.

Definition 1.2.3 (Verhulst, 1996)

An autonomous difference equation is one that does not explicitly include time as a variable.

Definition 1.2.4 (Verhulst, 1996)

A difference equation is called as linear if it can be expressed in the following form:

$$x_{n+k} + c_1(n)x_{n+k-1} + c_2(n)x_{n+k-2} + \cdots + c_k(n)x_n = R_n$$

where $c_i(n), i = 1,2, \dots, k$ and R_n are given functions of n.

Definition 1.2.5 (Kelley & Peterson, 2001)

A function that meets the criteria of the specified difference equation is considered a solution to that equation. A number of arbitrary constants that match the equation's order are included in the general solution of the difference equation. However, by giving these arbitrary constants definite values, a specific solution can be obtained from the general solution.

Definition 1.2.6 (Kelley & Peterson, 2001)

The product of a sequence of numbers $\{j_1, j_2, \dots, j_m\}$ is defined recursively, in a manner similar to that of summation.

- If the sequence contains only one number, the product is simply that number:

$$\prod_{i=1}^{1} j_i = j_1$$

- If the sequence contains more than one element, the product is defined by multiplying the product of the first $m-1$ elements by the m^{th}element:

$$\prod_{i=1}^{m} j_i = (\prod_{i=1}^{m-1} j_i) j_m.$$

Definition 1.2.7(Shoukaku, 2017)

A solution $\{x_m\}$ to equation (E) is called to be oscillatory if it changes sign infinitely often and does not stabilize to a constant sign beyond some fixed index.

Definition 1.2.8 (Shoukaku, 2017)

(a) A difference equation (E) is said to be oscillatory if all of its solutions are oscillatory.

(b) If at least one solution is non-oscillatory, the equation is classified as non-oscillatory.

Definition 1.2.9 (Agarwal, et al., 2005)

An equation (E) is said to be super-linear if

$$0 < \int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < \infty$$

for all $\varepsilon > 0$.

Definition 1.2.10 (Agarwal et al., 2005)

An equation (E) is said to be sub-linear if

$$0 < \int_{0}^{\pm\varepsilon} \frac{du}{f(u)} < \infty$$

for all $\varepsilon > 0$.

Definition 1.2.11 (Agarwal et al., 2005)

An equation (E) is said to be mixed type if

$$0 < \int_{\pm 0}^{\pm\infty} \frac{du}{f(u)} < \infty$$

Definition 1.2.12 (Agarwal et al., 2005)

- An equation (E) is highly oscillatory, if the equation

$$\Delta(c_m \Delta x_m) + \theta_m\, \varphi(x_m, \Delta x_m) + \gamma q_{m+1} f(x_{m+1}) = 0, m \in \mathbb{N}$$

  is oscillatory for all $\gamma > 0$.

- The equation (E) is classified as strongly non-oscillatory if the last equation is non-oscillatory for all $\gamma > 0$.

- The equation (E) is classified as conditionally oscillatory if the last equation is oscillatory for all $\gamma > 0$.

Definition 1.2.13 (Agarwal et al., 2000)

The sequence $\{x_m\}_{m=1}^{\infty}$ is called oscillatory about $c(c \in \mathbb{R})$ if an increasing sequence of integers exists $\{l_k\}_{k=1}^{\infty}$ such that

$$\left(x_{l_k} - c\right)\left(x_{l_{k+1}} - c\right) \leq 0 \text{ for all } k \in \mathbb{N}.$$

Definition 1.2.14 (Agarwal et al., 2000)

The sequence $\{x_m\}_{m=1}^{\infty}$ is defined as strictly oscillatory about 0, if for every $m \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $x_m x_{m+k} < 0$, without any ambiguity.

Definition 1.2.15 (Agarwal et al., 2000)

The sequence $\{x_m\}_{m=1}^{\infty}$ is called oscillatory about the sequence $\{y_m\}_{m=1}^{\infty}$ if an increasing sequence of positive integers exists $\{l_k\}_{k=1}^{\infty}$ such that

$$\left(x_{l_k} - y_{l_k}\right)\left(x_{l_{k+1}} - y_{l_{k+1}}\right) \leq 0 \text{ for all } k \in \mathbb{N}.$$

From the aforementioned definition it is evident that if the sequence $\{x_m\}_{m=1}^{\infty}$ oscillates about the sequence $\{y_m\}_{m=1}^{\infty}$ then $\{y_m\}_{m=1}^{\infty}$ oscillates about the sequence $\{x_m\}_{m=1}^{\infty}$. Moreover, every sequence $\{x_m\}_{m=1}^{\infty}$ oscillates about itself.

Definition 1.2.16 (Agarwal et al., 2000)

The sequence $\{x_m\}_{m=1}^{\infty}$ is called periodic (τ-period) if there exists an integer $\tau \in \mathbb{N}$ such that $x_{m+\tau} = x_m$ for all $m \in \mathbb{N}$.

Definition 1.2.17 (Agarwal et al., 2000)

A sequence $\{x_m\}_{m=1}^{\infty}$ is called periodic oscillatory if the sequence $\{sgn(x,m)\}_{m=1}^{\infty}$ is periodic, where

$$sgn(x,m) = \begin{cases} 1 & if\ x(m) > 0 \\ -1 & if\ x(m) < 0 \\ 0 & if\ x(m) = 0. \end{cases}$$

Definition 1.2.18 (Abbotts, 2015)

A sequence is defined as a function whose domain consists of the positive integers $\mathbb{N}$, or sometimes the nonnegative integers. The output of the function at a specific index m is represented by a_m , and the entire sequence is usually written as $\{a_m\}$ or (a_m). Therefore, a sequence can be understood as an ordered list of elements $a_1, a_2, a_3,\ldots$ generated based on a rule established by the defining function.

Definition 1.2.19 (Abbotts, 2015)

A sequence $\{a_m\}$ is said to converge to a real number L in ℝ if, for every $\mathcal{E} > 0$, there exists a positive integer N ∈ ℕ such that for all $m \geq N$, the inequality

$$|a_m - L| < \varepsilon$$

holds. In this case, L is called the limit of the sequence, and we write $\lim_{m \to \infty} a_m = L$.

Definition 1.2.20 (Abbotts, 2015)

A sequence $\{a_m\}$ is said to diverge if it does not approach any finite real number as m tends to infinity.

1.3 Properties of products (Kelley & Peterson, 2001)

Let j_i, k_i, and v be real (or complex) numbers, and let m be a constant. The following identities are then true for a positive integer m:

1. $\prod_{i=1}^{m}(j_i k_i) = (\prod_{i=1}^{m} j_i)(\prod_{i=1}^{m} k_i)$.
2. $\prod_{i=1}^{m}(j_i^l) = (\prod_{i=1}^{m} j_i)^l$.
3. $\prod_{i=1}^{m} v = v^m$.

The proof of identity 1, by mathematical induction, we get

Step 1: Base case ($m = 1$)

$$\prod_{i=1}^{1}(j_i k_i) = \left(\prod_{i=1}^{1} j_i\right)\left(\prod_{i=1}^{1} k_i\right) = j_1 k_1$$

Thus, for $m = 1$, the identity is true

Step 2: Assume that the identity is true for some $m \geq 1$, i.e.,

$$\prod_{i=1}^{m}(j_i k_i) = \left(\prod_{i=1}^{m} j_i\right)\left(\prod_{i=1}^{m} k_i\right)$$

Step 3: We want to show it holds for $m + 1$. Then:

$$\prod_{i=1}^{m+1}(j_i k_i) = \prod_{i=1}^{m}(j_i k_i) \, . \, (j_{m+1} k_{m+1})$$

According to the inductive hypothesis:

$$= \prod_{i=1}^{m} j_i \prod_{i=1}^{m} k_i (j_{m+1} k_{m+1})$$

Making use of multiplication's commutatively and associativity

$$= \left(\prod_{i=1}^{m} j_i\, j_{m+1}\right)\left(\prod_{i=1}^{m} k_i\, k_{m+1}\right)$$

$$= \prod_{i=1}^{m+1} j_i \prod_{i=1}^{m+1} k_i$$

1.4 Difference Operator Δ

The basic component of calculations of finite differences in difference calculus is the difference operator. Its role is similar to the differential operator in differential calculus.

Definition 1.4.1 (Kelley & Peterson, 2001)

The forward operator Δ is defined for any sequence $(\{x_m\}_{m=1}^{\infty}): \mathbb{N} \to \mathbb{R}$ of real numbers as follows $\Delta x_m = x_{m+1} - x_m$. This expression is referred to as the difference of x_m, and we denote Δ as the first difference operator.

Higher order differences are iterations of the basic difference operator. For example, the second order difference is

$$\begin{aligned}\Delta^2 x_m &= \Delta(\Delta x_m) = \Delta(x_{m+1} - x_m)\\ &= (x_{m+2} - x_{m+1}) - (x_{m+1} - x_m)\\ &= (x_{m+2}) - 2x_{m+1} + x_m.\end{aligned}$$

The following formula for the k^{th} order difference is readily verified by induction:

$$\begin{aligned}\Delta^k x_m &= x_{k+m} - m x_{k+m-1} + \frac{m(m-1)}{2!} x_{k+m-2} + \cdots + (-1)^m x_k\\ &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} x_{k+n-m}\end{aligned}$$

It can be shown that Δ is linear operator, i.e.

$$\Delta(c_1 x_m + c_2 y_m) = c_1 \Delta x_m + c_2 \Delta y_m$$

Where $c_1\ and\ c_2$ constants.

It is occasionally possible to apply the difference operator to functions that involve two or more variables. In these cases, the variable that needs to be increased by one unit will be indicated by a subscript. For example,

$$\Delta_t t\, e^m = (t+1)\, e^m - t\, e^m,$$

while

$$\Delta_m t\, e^m = t\, e^{m+1} - t\, e^m = t\, e^m(e-1).$$

Definition 1.4.2 (Kelley & Peterson, 2001**)**

The shift operator E is defined as $E\ (x_m) = x_{m+1}$.

It is easy to see that

$$E^k(x_m) = x_{m+k}.$$

Definition 1.4.3 (Sultana, 2015)

The identity operator denoted as I is defined by

$$Ix(m) = x(m).$$

The composition of I and E is the same as multiplication of numbers. Clearly $\Delta = E - I$.

Definition 1.4.4 (Kelley & Peterson, 2001**)**

The function $f: \mathbb{N} \to \mathbb{R}$ is an arithmetic sequence of order n if $\Delta^{n+1}(f)$ is identically zero, but $\Delta^n(f)$ is not.

Some the fundamental properties of Δ are given in the following theorem:

Theorem 1.4.1 (Sultana ,2015)

For any $m \in \mathbb{N}$ and any $a \in \mathbb{R}$:

1) $\Delta(ax_m) = a\Delta x_m$ if a is constant**.**

Proof.

$$\begin{aligned}\Delta(ax_m) &= ax_{m+1} - ax_m,\\ &= a(x_{m+1} - x_m)\\ &= a\Delta x_m.\end{aligned}$$

2) $\Delta(x_m + y_m) = \Delta(x_m) + \Delta(y_m)$.

Proof.

$$\begin{aligned}\Delta(x_m + y_m) &= x_{m+1} + y_{m+1} - (x_m + y_m)\\ &= x_{m+1} - x_m + y_{m+1} - y_m\\ &= \Delta(x_m) + \Delta(y_m).\end{aligned}$$

3) $\Delta(x_m y_m) = y_{m+1}\Delta(x_m) + x_m\Delta(y_m)$.

Proof.

$$\begin{aligned}\Delta(x_m y_m) &= x_{m+1}y_{m+1} - x_m y_m\\ &= x_{m+1}y_{m+1} - x_m y_{m+1} + x_m y_{m+1} - x_m y_m\\ &= y_{m+1}[x_{m+1} - x_m] + x_m[y_{m+1} - y_m]\\ &= y_{m+1}\Delta(x_m) + x_m\Delta(y_m).\end{aligned}$$

4) $\Delta\left(\frac{x_m}{y_m}\right) = \frac{y_m \Delta x_m - x_m \Delta y_m}{y_m y_{m+1}}$.

Proof.

$$\begin{aligned}\Delta\left(\frac{x_m}{y_m}\right) &= \frac{x_{m+1}}{y_{m+1}} - \frac{x_m}{y_m}\\ &= \frac{y_m x_{m+1} - x_m y_{m+1} - x_m y_m + x_m y_m}{y_m y_{m+1}}\\ &= \frac{y_m(x_{m+1} - x_m) - x_m(y_{m+1} - y_m)}{y_m y_{m+1}}\\ &= \frac{y_m \Delta x_m - x_m \Delta y_m}{y_m y_{m+1}}.\end{aligned}$$

Theorem 1.4.2. (Jordan, 1950)

Let α be any constant. Then

$$\Delta\alpha^m = (\alpha - 1)\alpha^m.$$

Proof.

$$\begin{aligned}\Delta\alpha^m &= \alpha^{m+1} - \alpha^m\\ &= (\alpha - 1)\alpha^m.\end{aligned}$$

Remark 1.4.1 (Sultana ,2015)

Some functions possess simple derivatives but can be difficult when calculating their differences. On the other hand, functions that are seldom covered in standard calculus can often be easily examined through difference methods. A key formula in differential calculus is the power rule, for instance.

$$d(x^m) = m\, x^{m-1}.$$

The difference of a power is not especially useful in difference calculus due to its complexity.

$$\begin{aligned}\Delta x^m &= (x+1)^m - x^m\\ &= \sum_{k=0}^{m} \binom{m}{k} x^k - x^m\\ &= \sum_{k=0}^{m-1} \binom{m}{k} x^k.\end{aligned}$$

1.5 Summation

To use the difference operator properly, we need to present an antidifference operator. This operator is referred to as the right inverse operator and is occasionally called the indefinite sum.

Definition 1.5.1 (Kelley & Peterson, 2001)

An indefinite sum of (n) , denoted by $\sum x(n)$, is any function such that

$$\Delta\left(\sum x(n)\right) = x(n)$$

for all n in the domain of x.

Theorem 1.5.1 (Kelley & Peterson, 2001)

The expression $\sum x(n) = y(n) + C(n)$, can be used to denote any indefinite sum of $x(n)$, if $y(n)$ is an indefinite sum of $x(n)$, where C fulfils $\Delta C(n) = 0$ and has the same domain as x.

Example 1.5.1 (Kelley & Peterson, 2001)

Determine the infinite sum $\sum 3^k$

Since,

$$\Delta\alpha^k = (\alpha - 1)\alpha^k$$

We have,

$$\Delta 3^k = 2.3^k$$

So,

$$\frac{\Delta 3^k}{2} = 3^k$$

$\frac{3^k}{2}$ is hence an unlimited sum of 3^k.

Assume that $\Delta C(k) = 0$ and that $C(k)$ is a function with the same domain as 3^k. Then,

$$\Delta\left(\frac{3^k}{2} + C(k)\right) = \Delta\left(\frac{3^k}{2}\right) = 3^k,$$

So, $\frac{3^k}{2} + C(k)$ is an indefinite sum of 3^k.

Additionally, if any infinite sum of 3^k. is represented by $g(k)$, then

$$\Delta\left(g(k) - \frac{3^k}{2}\right) = \Delta g(k) - \Delta\left(\frac{3^k}{2}\right) = 3^k - 3^k = 0.$$

So, $g(k) = \frac{3^k}{2} + C(k)$ for some $C(k)$ with $\Delta C(k) = 0$. It follows that we have found all indefinite sums of 3^k, and we write

$$\sum 3^k = \frac{3^k}{2} + C(t),$$

Where $C(t)$ is any function with the same domain as 3^k and $\Delta C(k) = 0$.

Corollary1.5.1 (Kelley & Peterson, 2001)

If the domain is represented by the set {a, a + 1, a + 2, ...}, every such indefinite sum will have the following form, where a can be any real number

and represents an indefinite sum of a with C being an arbitrary constant,

$$\sum x(n) = y(n) + C.$$

From Theorem 1.5.1, we can obtain several basic characteristics of the indefinite sum.

Theorem 1.5.2 (Kelley & Peterson, 2001)

For any constant c

a) $\sum\big(x(n) + y(n)\big) = \sum x(n) + \sum y(n)$.

b) $\sum\big(cx(n)\big) = c \sum x(n)$.

c) $\sum\big(x(n)\Delta y(n)\big) = x(n)y(n) - \sum Ey(n)\Delta x(n)$.

d) $\sum\big(Ex(n)\Delta y(n)\big) = x(n)y(n) - \sum y(n)\Delta x(n)$.

Remark 1.5.1 (Sultana,2015) Just as integration by parts is employed to evaluate integrals, summation by parts can be utilized to derive various forms of indefinite sums. Additionally, these formulas are essential for the analysis of difference equations. Lastly, the corollary 1.6.1 provides us with

$$\sum x_n = \sum_{k=m}^{n-1} x_k + C \tag{1.2}$$

for some constant C and, alternatively, for $n \le p$,

$$\sum x_n = -\sum_{k=n}^{p} x_k + C$$

(1.3)

for some constant C.

Equations (2.2) and (2.3) provide us with a method to connect indefinite sums to definite sums.

The following theorem, used to calculate definite sums that are similar to the Fundamental theorem of calculus.

Theorem 1.5.3 (Fundamental Theorem of Difference Calculus) (Kelley & Peterson, 2001)

If $y(n)$ is an indefinite sum of $x(n)$, then for $m < n$

$$\sum_{k=m}^{n-1} y(k) = [x(k)]_m^n = x(n) - x(m).$$

Proof.

For $m < n$, we obtain x_n, which is the indefinite sum of $y(n)$

$$\Delta x_n = x_{n+1} - x_n = y_n.$$

This implies:

$$y_k = \Delta x_k = x_{k+1} - x_k$$

Summing y_k from $k = m$ to $k = n - 1$, we get

$$\sum_{k=m}^{n-1} y_k = \sum_{k=m}^{n-1} \Delta x_k = \sum_{k=m}^{n-1} (x_{k+1} - x_k)$$

$$\sum_{k=m}^{n-1} (x_{k+1} - x_k) = (x_{m+1} - x_m) + (x_{m+2} - x_{m+1}) + \cdots + (x_n - x_{n-1}).$$

Subsequent to cancelation $x_n - x_m$.

Therefore

$$\sum_{k=m}^{n-1} y_k = x_n - x_m.$$

Theorem 1.5.4 (Kelley & Peterson, 2001)

If $m < n$, then

$$\sum_{k=m}^{n-1} \alpha_k \Delta\beta_k = [\alpha_k \beta_k]_m^n - \sum_{k=m}^{n-1} (\Delta\alpha_k)\beta_{k+1}.$$

Proof.

Let $x_n = \alpha_k$ and $y_n = \beta_k$, then by

$$\sum (x_n \, \Delta y_n) = x_n y_n - \sum E y_n \Delta x_n$$

we have

$$\sum(\alpha_n\Delta\beta_n)=\alpha_n\beta_n-\sum(\Delta\alpha_n)\beta_{n+1}.$$

By

$$\sum y_n=\sum_{k=m}^{n-1}y_k+C$$

$$\sum_{k=m}^{n-1}(\alpha_k\Delta\beta_k)=[\alpha_k\beta_k]_m^n-\sum_{k=m}^{n-1}(\Delta\alpha_k)\beta_{k+1}+C\,.$$

Substituting $n = m + 1$ in the above equation, we get

$$\sum\alpha_m\Delta\beta_m=\alpha_{m+1}\,\beta_{m+1}-(\Delta\alpha_m)\beta_{m+1}+C.$$

This implies $C = -\alpha_m\beta_m$, and hence, the proof is complete.

Remark 1.5.2 (Sultana,2015)

An equivalent form of Theorem 1.6.4 is Abel's summation formula

$$\sum_{\boldsymbol{k=m}}^{\boldsymbol{n-1}}\alpha_k\beta_k=\beta_n\sum_{k=m}^{n-1}\alpha_k-\sum_{k=m}^{n-1}\left(\sum_{i=m}^{k}\alpha_i\right)\Delta\beta_k,\quad(n>m)$$

and, alternatively,

$$\sum_{\boldsymbol{k=n}}^{\boldsymbol{p}}\alpha_k\beta_k=\beta_{n-1}\sum_{k=m}^{n-1}\alpha_k+\sum_{k=n}^{p}\left(\sum_{i=k}^{p}\alpha_i\right)\Delta\beta_{k-1},\quad(p\ \geq\ n).$$

Lemma1.5.1 (Kelley & Peterson, 2001)

The following statement is satisfied

$$\sum_{k=n_0}^{n-1}\Delta x(k)=x(n)-x(n_0).$$

Proof.

$$\sum_{k=n_0}^{n-1}\Delta x(k)=\sum_{k=n_0}^{n-1}\left[\sum_{i=0}^{1}(-1)^i\binom{1}{i}x(k+1-i)\right]$$

$$=\sum_{k=n_0}^{n-1}[x(k+1)-x(k)]$$

$$=x(n)-x(n_0).$$

1.6 Riccati transformation

The Riccati transformation has emerged as a fundamental analytical tool in the qualitative study of differential and difference equations, particularly in the context of oscillation theory. By converting second-order equations into first-order nonlinear Riccati-type equations, this transformation enables a deeper understanding of the oscillatory behavior of solutions through comparison techniques, integral inequalities, and spectral analysis. Specifically, for a second-order linear differential equation of the form $y'' + q(x)y = 0$, the substitution $r(x) = y'(x)/y(x)$ leads to the Riccati equation $r'(x) + r(x)^2 + q(x) = 0$ which encapsulates the essential features of the original system in a more tractable form.Recent research has demonstrated the versatility and effectiveness of Riccati-based methods in establishing new oscillation criteria across various classes of equations. For instance, Erbe and Kong (2021) applied Riccati techniques to second-order linear difference equations, deriving sharp conditions for oscillation using nonlinear comparison arguments. Similarly, Ahmad and El-Sayed (2020) developed novel oscillatory criteria for fractional differential equations, highlighting the role of Riccati-type inequalities in fractional calculus. In the realm of delay equations, Al-Khateeb and Al-Sarayreh (2024) investigated fractional difference equations with delay terms, where Riccati transformations facilitated the analysis of zero distributions and solution behavior. Moreover, Hooker and Wong (1987) provided a foundational framework for Riccati methods in linear systems, demonstrating their power in extending classical oscillation results to matrix-valued and system-level equations.

In my master's thesis, I have adopted a similar approach by employing the Riccati transformation to establish my own oscillation theorems. This method was chosen due to its significant analytical power in studying the oscillatory behavior of solutions to difference equations, allowing for a more precise characterization of zero distributions and qualitative dynamics.

These contributions underscore the central role of the Riccati transformation in modern oscillation theory, offering a unified approach to studying the qualitative dynamics of differential, difference, and fractional systems. The present work builds upon these insights to further explore the application of Riccati techniques in proving oscillation theorems for second-order differential equations with variable coefficients and delay structures.

Chapter Two

Literature Review

2.1 Introduction

This chapter focuses on the study of the most relevant previous work for the oscillation of second order difference equations, especially the damping term, in relation with the problem that we address. The overview will specify the basic work in this field, including the key equations which improve upon previous results. It is of interest to have a good understanding of oscillatory phenomena for the solutions of second-order difference equations with a nonlinear damping term. There is a great urgency to search for global explanations on this trend, which led many researches to carry out in-deep investigations, with the aim of characterizing the regions where such oscillations occur. Several approaches have been used to analyzes the oscillations, and the various models proposed have provided useful information. In this part, a few studies relevant to the subject of the current investigation are cited. Such researches have provided the theoretical basis of the study.

2.2 Literature review

The examination of oscillatory behavior in difference equations has attracted considerable interest over the years. The fundamental work commenced with Atkinson (1964), who presented the inaugural oscillation criteria for second-order linear difference equations, employing concepts of stability and comparison theory.

In the early 1980s, Hooker and Patula (1983) extended the oscillation analysis to nonlinear cases. They demonstrated that the nonexistence of monotonic solutions implies that all solutions must oscillate. In the same year, Szmanda (1983) investigated second-order nonlinear difference equations of the form

$$\Delta^2 x_n + \sum_{i=1}^{m} a_{in}\, f_i(x_n, \Delta\, x_n) = 0, \qquad n = 0,1,2,\dots \tag{2.1}$$

The following conditions are assumed to hold

a) $a_{in} \geq 0$ for $n \geq n_0 \geq 0,\ i = 1,\dots,m$;

b) $f_i\colon \mathbb{R}^2 \to \mathbb{R} = (-\infty,\infty), x f_i(x,y) > 0$ for $x \neq 0, i = 1,\dots,m$.

and suppose there exists an index k such that

$f_k(x,y)$ is continuous on $\mathbb{R}^2$ and

$$\sum_{k}^{\infty} n a_{kn} = \infty.$$

Then, all of the equation is solutions are oscillatory.

Miller (1985) advanced the analysis of difference equations by developing the Riccati transformation technique, transforming the original equation into a new form from which oscillation terms can be easily derived, opening the door to the creation of more efficient techniques, while Miller (1985) relied on classical Riccati transformations for linear damping, our modified transform handles nonlinear $\varphi(y_n, \Delta y_n)$ with weaker convergence conditions.

In the 1990s, Thandapani (1995) concentrated on second-order nonlinear difference equations characterized by the form

$$\Delta(a_n \Delta x_n) + p_n \Delta x_n + q_n f(x_{n+1}) = 0\,, n = 0,1,2, \dots \tag{2.2}$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers and the real sequences $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ and the function f satisfy the following conditions

a) $a_n > 0, p_n \geq 0$ and $q_n > 0$ for all $n \geq n_0 \geq 0$;
b) The function $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $uf(u) > 0$ for all $u \neq 0$.

Lemma 2.2.1 (Thandapani, 1995)

Suppose that $a_n - p_n > 0$ For any large n, a nd

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \prod_{s=n_0}^{n-1} \left(1 - \frac{p_s}{a_s}\right)\right) = \infty.$$

Since $\{x_n\}$ is a non-oscillatory solution of equation (2.1), then for any $n \geq N$. there exists $N \geq 0$ such that $x_n \Delta x_n > 0$.

Remark 2.2.1

Equation (2.2) is a specific case of Equation (E) when $\varphi(x_n, \Delta x_n) = \Delta x_n$, and the coefficient q_{n+1} is replaced by q_n. The research provided initial data that served as the foundation for many subsequent studies that expanded upon or refined its findings.

Although Thandapani's research focused solely on the analysis of linear damping equations involving $p_n \Delta x_n$ under specific constraints on (x), our study expands this by substituting the linear damping coefficient with a more general nonlinear function, $\varphi(x_n, \Delta x_n)$.

Thandapani and Lalli (1995) made additional contributions by proposing a more generalized equation.

$$\Delta^2 x_n + p_n \Delta x_n + q_n f(x_{n+1}) = 0 \,, n \in \mathbb{N},$$

(2.3)

where

a) For an unlimited number of values of n, q_n is nonzero in the real-valued sequences $\{p_n\}$ and $\{q_n\}$.
b) The function $f : \mathbb{R} \to \mathbb{R}$ continues and fulfils $xf(x) > 0$ for each $x \in \mathbb{R} \backslash \{0\}$.
c) For any $u \neq o, v \neq 0$ and $g(u,v) \geq L > 0$ there exists a function of real value g such that that $f(u) - f(v) = g(u,v)(u-v)$.

Theorem 2.2.1 (Thandapani & Lalli,1995)
Suppose that conditions a, b and c hold, assume that
$(n+1)p_n \leq 1$, for each sufficiently significant amount of n,

$$\sum_{n}^{\infty} (n+1){p_n}^2 < \infty,$$

$$\sum\nolimits_{n=n_0}^{\infty} (n+1) q_{n+1} = \infty,$$

$$\int_{\varepsilon}^{\infty} \frac{du}{f(u)} < \infty,$$

$$\int_{-\infty}^{-\varepsilon} \frac{du}{f(u)} > -\infty,$$

for $\varepsilon > 0$ hold. then equation (2.3) is oscillatory.

Consequently, all solutions to the equation manifest oscillatory behavior. The researchers formulated three theories that outline oscillation criteria applicable to various situations, underscored by examples that supported the effectiveness for these criteria. They constitute a pioneering endeavor to expand the scope of oscillation criteria, encompassing cases that extend beyond specific coefficient signals. It serves as a valuable reference for the evaluation of damped nonlinear difference equations.

Thandapani and Pandian (1995) made a significant contribution to the field by further generalizing the existing framework through the incorporation of additional nonlinear terms, thereby enhancing the analytical depth of oscillation criteria for second-order difference equations

$$\Delta^2 x_n + p_n \varphi(x_n, \Delta x_n) + q_n f(x_{n+1}) g(\Delta x_n) = 0\ , n = 0,1,2, \ldots \qquad (2.4)$$

This covers older types as exceptional circumstances. They placed generalized growth and monotonicity restrictions on the nonlinear functions φ, f, and g and demonstrated oscillation criteria utilizing advanced comparison and integral test techniques.

Remark 2.2.2

In particular, Equation (2.4) generalizes Equation (2.3), which can be obtained as a specific case by $\varphi(x_n, \Delta x_n) = \Delta x_n$ and $g(\Delta x_n) = 1$.

Theorem 2.2.2 (Thandapani & Pandian, 1995)

Assuming that conditions

a) $p_n \geq 0$ and , $q_n > 0\ \ \forall n \in \mathbb{N}$, and such that

$$1 - M p_n > 0\ \ \forall\, n \geq N_0 \in \mathbb{N}$$

and

b) $\sum_{n=N_0}^{\infty} \left[\prod_{s=N_0}^{n-1} (1 - M p_s) \right] = \infty$

c) For every $y \neq 0$ and $x \in \mathbb{R}$, there is a constant $M > 0$ such that $0 < \varphi(x, y) x \leq M x^2$.

d) For any $x \neq 0$, then $x\, f(x) > 0$ and $g(x) > 0$.

e) Assume that f is non-decreasing, and it satisfies the inequality $f(-x_1 x_2) \geq f(x_1 x_2) \geq K f(x_1) f(x_2)$ for $x_1, x_2\ > 0$, for some constant K,

f) and assume that for any $u \neq 0$, $f(u)$ and , $g(u)$ are non-decreasing;

$$\int_{+0} \frac{du}{f(u) g(u)} < \infty, \qquad \int_{-0} \frac{du}{f(u) g(u)} < \infty$$

and

$$\sum_{n}^{\infty} f(n) q_n = \infty.$$

Hence, all solutions to the equation (2.4) are oscillatory.

Both equations (2.4) and (E) are second-order nonlinear difference equations, but they possess distinct structural features. Equation (2.4) presents a more generalized nonlinear form by including both $\varphi(x_n, \Delta x_n)$ and the multiplicative term $g(\Delta x_n)$, thereby increasing its analytical complexity. In contrast, Equation (E) provides a simpler structure by omitting the function g, facilitating a more straightforward analysis through classical methods. Moreover, Equation (E) features a forward shift in the index of the coefficient q, which could influence the quality of its solutions.

Zhang and Ping (2001) the objective of this study is to examine the oscillation behavior of solutions to second-order nonlinear difference equations of the form.

$$\left.\begin{array}{l} \Delta^2 x_n + a_n \Delta x_n + b_n f(y_{n-\tau}) = 0 \\ \\ \Delta^2 x_n + a_n \Delta x_{n+1} + b_n f(y_{n-\tau}) = 0, \end{array}\right\} \qquad (2.5)$$

where τ is a positive integer, the sequences $\{a_n\}$ and $\{b_n\}$ are real-valued and the function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

From this standpoint, equation (E) functions as an all-encompassing framework for investigating the dynamic behavior of nonlinear difference equations, encompassing those that feature delay elements, anticipation, or intricate nonlinear interactions. This structural versatility is a key characteristic of the model, allowing for generalization and a unified analysis across a diverse array of distinct models.

Raslan and Salem (2008) The following equation of the form are the main focus of the study:

$$\Delta(r_n \Delta x_n) + p_n \Delta x_n + q_n f(x_{n+1}) = 0, \quad n \in \mathbb{N}, \qquad (2.6)$$

where

a) The real-valued sequences $\{r_n\}, \{p_n\}$ and $\{q_n\}$ are not all equal to zero for an unlimited number of values.
b) For any $x \neq 0$, Functions $f, \psi: \mathbb{R} \to \mathbb{R}$ are continuous.
c) For any $u \neq o, v \neq 0$, and $g(u,v) \geq L > 0$ there exists a real-valued function g such that $f(u) - f(v) = g(u,v)(u - v)$.

Conditions such as:

d) $s^2 = \sum_{n=M}^{\infty}(n+1)p_n^2 < \infty; \quad s > 0,$

e) $\sum_{n=0}^{\infty} r_n^2 < \infty,$

f) $\sum_{n=0}^{\infty}(n+1)q_n = \infty,$

g) $(n+1)p_n \leq r_n$ for sufficiently large n

and

$(n+1)p_n \leq r_n\psi(x_n)$ for sufficiently large *n* are used to derive oscillation results.

The paper employs mathematical methods based on contradiction techniques, thereby proving several theorems by demonstrating inconsistencies. The objective is to demonstrate that a solution is non-oscillatory (i.e. it ultimately maintains a single sign) and subsequently formulate conditions that lead to a contradiction. This is done in order to show that all solutions must oscillate. Furthermore, the paper employs the Schwarz inequality to address expressions of the form.

$$\left(\sum a_n b_n\right)^2 \leq \left(\sum a_n{}^2\right)\left(\sum b_n{}^2\right)$$

The present study bears a resemblance to the work of Sh. Salem and K.R. Raslan (2008), yet it predates the aforementioned research and introduces conditions on the sequence $\{a_n\}$ to ensure that the solution remains oscillatory, even in the presence of damping terms.

Theorem 2.2.3 (Salem & Raslan, 2008)

Suppose that (a)- (g) are hold, then all solutions to equation (2.6) will be oscillatory.

Conversely, equation (2.6) has a simpler structure, which often facilitates easier analysis using traditional methods. This is especially beneficial when the system displays linear or only slightly nonlinear behavior, or when analytical simplicity is a key concern. In conclusion, while equation (E) is appropriate for broader theoretical studies, the simpler form is more suitable for practical applications or illustrative purposes.

Dogan and Bolat (2021) The following equation of the form are the main focus of the study:

$$\Delta_{l,a}\left(p_n\Delta_{l,a}x_n\right) + q_n\left(\Delta_{l,a}x_n\right)^{\beta} = F\left(n, x_n, \Delta_{l,b}x_n\right), \quad n \in \mathbb{N}, \tag{2.7}$$

Where $\Delta_{l,\sigma}$ is generalized difference operator such that defined as $\Delta_{l,\sigma}x_n = x_{n+l} - \sigma x_n$, and $F\colon \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}$.

Theorem 2.2.4 (Dogan & Bolat, 2021)

Let $a < 0, 0 < \beta < 1$ such that $\frac{2}{\beta - 1}$ is the ratio of two odd integers, $p_n > 0$, $q_n > 0$ and the condition

$$\begin{cases} F(n,u,v) = 0, & v + (b-a)u \neq 0, \quad n \in \mathbb{N}, u, v \in \mathbb{R} \\ \dfrac{v + (b-a)u}{p_{n+l}} F(n,u,v) \leq 0, & v + (b-a)u \neq 0, \quad n \in \mathbb{N}, u, v \in \mathbb{R} \end{cases}$$

is satisfied, then all nontrivial solutions of (2.7) are oscillatory.

The equation under investigation represents a significant advancement beyond the second-order difference equations explored in earlier research, particularly those addressing oscillatory behavior influenced by damping terms. The theoretical foundation for this advancement has been laid in previous studies. This research aims to extend existing oscillation results by utilizing similar analytical techniques such as comparison principles and arguments by contradiction within a more generalized context as represented by equation (E). Accordingly, the study not only builds upon earlier contributions but also seeks to establish a more comprehensive analytical framework for analyzing the behavior of solutions to nonlinear difference equations with intricate damping structures.

Chapter Three

Analytical Comparison between the results of (Thandapani,1995) and (Thandapani & Pandian,1995)

3.1 Introduction

This chapter presents an analytical comparison between two influential studies in the field of oscillation theory: Thandapani (1995) and Thandapani & Pandian (1995). Both works contribute significantly to the development of oscillation criteria for nonlinear difference equations, yet they differ in their methodological frameworks, assumptions, and the scope of their results. The purpose of this comparative analysis is to highlight the evolution of techniques used, particularly the role of Riccati transformations and comparison theorems, and to examine how the generalizations introduced in the latter study build upon or diverge from the foundational results of the former.

By systematically evaluating the conditions under which oscillatory behavior is guaranteed, this chapter aims to clarify the strengths and limitations of each approach. Special attention is given to the nature of the nonlinearities considered, the structure of the difference equations analyzed, and the implications of the derived criteria. This comparison not only underscores the progression of analytical tools in the study of discrete dynamical systems but also provides a contextual foundation for the results developed in the present thesis.

3.2 Analytical Comparative

The researcher by Ethiraju Thandapani, 1995 Published in Czechoslovak Mathematical Journal, Vol. 45, focuses on the study of the damped nonlinear difference equation of the form:

$$\Delta(a_n \Delta y_n) + p_n \Delta y_n + q_n f(y_{n+1}) = 0 \ , n = 0,1,2, \ldots \qquad (3.1)$$

The main aim is to establish new oscillation criteria and provide sufficient conditions that ensure that all solutions of equation (3.1) are oscillatory.

Damping term in equation (3.1) is $p_n \Delta y_n$, this limit is linear concerning the first difference Δy_n, which means that the damping effect is directly proportional to the speed of change of the sequence, but Nonlinearity is introduced into the equation through the function $f(y_{n+1})$.

While The researchers Thandapani and Pandian published in Tamkang Journal of Mathematics, Vol. 26 (1995), propose a more general formulation of the second-order nonlinear difference equation:

$$\Delta^2 y_n + p_n \varphi(y_n, \Delta y_n) + q_n f(y_{n+1}) g(\Delta y_n) = 0 \, , n = 0,1,2, \ldots \quad (3.2)$$

The aim of Thandapani and Pandian (1995) is identical to the first research's aim to provide sufficient conditions to ensure that all solutions of the equation are oscillatory. The crucial difference lies in the damping term $p_n \varphi(y_n, \Delta y_n)$ which is explicitly defined to be nonlinear. It depends not only on Δy_n but may also depend on y_n, which represents an important generalization of the equation (3.1). Also, we note the nonlinearity here is more complex, appearing in one damping term via the function φ and in the other term via the functions f and g.

This chapter discusses and compares only the theories that were addressed in the results section of our study.

Theorem 3.2.1 In addition to conditions

1) $a_n - p_n > 0$ for all $n \geq n_0 \geq 0$

2) $\sum^{\infty} \frac{1}{a_n} \left[\prod_{s=0}^{n-1} \left(1 - \frac{p_s}{a_s} \right) \right] = \infty$

3) $\int^{+\infty} \frac{du}{f(u)} < \infty$ and $\int^{-\infty} \frac{du}{f(u)} < \infty$

and

4) $f(u) - f(v) = g(u, v)(u - v)$, $\; g(u, v) \geq M > 0$ for $u, v \neq 0$.

assume that there is a constant $K > 0$ and a positive nondecreasing sequence $\{h_n\}$ such that

5) $\Delta(a_{n+1} \Delta h_n) \leq 0$ and $p_n \geq -\frac{K}{h_n}$ for all $n \geq N > 0$

If

6) $\sum^{\infty} h_n q_{n+1} = \infty$

Then all solutions of (3.1) are oscillatory.

Example 3.2.2 Examine the difference equation:

$$\Delta^2 y_n - \frac{1}{n+3}\Delta y_n + \frac{(4n^2+10n+5)}{n(n+5)^3}(y_{n+1}^3) = 0, \quad n \geq 1.$$

All conditions of Theorem (3.3.1) are verified. Hence every solution of this equation are oscillatory.

Theorem 3.2.3 Suppose that (1), (2) hole. Assume that

7) $\int^{\pm c} \frac{du}{f(u)} < \infty \quad \forall c > 0,$

8) f does not decrease and $f(xy) \geq \gamma f(x)f(y)$ and $-f(-xy) \geq \gamma f(x)f(y)$ for some constant $\gamma > 0, x \text{ and } y \ \in \mathbb{R}\backslash\{0\}$

and

9) $\sum^{\infty} f\big(R(n)\big)q_{n+1} = \infty, \quad R(n) = \sum_{s=n_0}^{n} \frac{1}{a_s}$

Then every solution of eq. (3.1) is oscillatory.

Example 3.2.4 Consider the following difference equation

$$\Delta\big((n+3)\Delta y_n\big) + \frac{1}{n+3}\Delta y_n + \frac{(4n^2+10n+7)}{n(n+5)^{\frac{7}{5}}}\left(y_{n+1}^{\frac{1}{5}}\right) = 0, \quad n \geq 1.$$

We note that, from Theorem (3.2.3) this equation is oscillatory.

Theorem 3.2.5 In addition to (1), (2), (3) and (6) assume that there exists a positive sequence $\{h_n\}$ such that

10) $(\Delta h_n) \geq 0$ and $\Delta(a_n \Delta h_n) \leq 0$ for all $n \geq n_0 \geq 0$.

Then every solution of Eq. (3.1) is oscillatory.

Example 3.2.6 Suppose that

$$\Delta\big((n+3)\Delta y_n\big) + \frac{1}{n+3}\Delta y_n + (n+3)(4n^2+10n+7)(y_{n+1}^3) = 0, \quad n \geq 1.$$

Based on Theorem (3.2.5) all solutions of this equation are oscillatory.

Theorem 3.2.7 Suppose that

1) $p_n \geq 0$ and $q_n > 0$ for all $n \in \mathbb{N}$;

2) There exists a constant $M > 0$ such that $0 < \varphi(x, y)y \leq My^2$ for all $y \in \mathbb{R}/\{0\}$ and $x \in \mathbb{R}$;

3) $yf(y) > 0$ and $g(y) > 0$ for all $y \neq 0$

4) f is non-decreasing and

$f(-xy) \geq f(xy) \geq K_1 f(x)f(y)$ for $x, y > 0$, some constant $K_1 > 0$;

and

5) $1 - MP_n > 0$ for all $n \in \mathbb{N}$,

6) $\sum_{n=N}^{\infty}[\prod_{s=N}^{n-1}(1 - Mp_s)] = \infty$.

7) $f(u)g(u)$ is nondecreasing for $u \neq 0$,

8) $\int_{+0}^{\alpha} \frac{d(u)}{f(u)g(u)} < \infty$ and $\int_{-\alpha}^{0} \frac{d(u)}{f(u)g(u)} > -\infty$ for all $\alpha > 0$ and

9) $\sum_{n=n_0}^{\infty} q_{n+1} f(n) = \infty$.

Then every solution of equation (3.2) is oscillatory.

Example 2.3.8 Consider the difference equation

$$\Delta^2 y_n + \frac{1}{n}\frac{\Delta y_n}{1+|\Delta y_n|} + \frac{6(10n+1)}{3n(1+n^{\frac{1}{5}})} f(y_{n+1})g(\Delta y_n) = 0, \quad n \geq 2.$$

Then every solution of Eq. (3.2) is oscillatory.

Theorem 3.2.9 Suppose that (1)- (3) and (5)- (7) hold, and assume that

10) g is non-decreasing and

$g(-xy) \geq g(xy) \geq K_2 g(x)g(y)$ for $x, y > 0$, some constant $K_2 > 0$;

11) $\int_{\alpha}^{\infty} \frac{d(u)}{f(u)g(u)} < \infty$ and $\int_{-\infty}^{-\alpha} \frac{d(u)}{f(u)g(u)} > -\infty$ for all $\alpha > 0$ and

12) $\sum_{n=N}^{\infty} nq_{n+1} g\left(\frac{2}{n}\right) = \infty$.

Then all solutions of Eq. (3.2) are oscillatory.

Example 3.2.10 Suppose that

$$\Delta^2 y_n + \frac{1}{n}\frac{\Delta y_n}{2+(\Delta y_n)^2} + \frac{(10n-1)}{3n}(y_{n+1}^3) = 0, \quad n \geq 1.$$

Then every solution of Eq. (3.2) is oscillatory.

We can summarize the comparison between the two studies in the following table:

(Thandapani, 1995)	**Aspect**	**(Thandapani & Pandian, 1995)**
$\Delta(a_n \Delta y_n) + p_n \Delta y_n + q_n f(y_{n+1}) = 0$	Equation	$\Delta^2 y_n + p_n \varphi(y_n, \Delta y_n) + q_n f(y_{n+1}) g(\Delta y_n) = 0$
General: a_n variable	Leading Term	Simplified: a_n =1
Linear: $p_n \Delta y_n$	Damping Term	Nonlinear: $p_n \varphi(y_n, \Delta y_n)$
f is non-decreasing, monotonic	Assumption	Conditions on φ and *g; f* and *g* are non-decreasing
Simpler systems with linear damping	Applications	More complex systems with nonlinear damping

The two papers represent two complementary contributions to the study of oscillation theories, the first, published alone by Thandapani, lays the foundation by studying the linearly damped difference equation. The second, in collaboration with Pandian, builds on this foundation and generalizes the results to a wider class of nonlinear damping equations, expanding the scope of application of these theories and providing a deeper understanding of oscillatory behavior in more complex systems. They can be considered two successive works that illustrate the development and extension for oscillation theory of difference equations.

Chapter Four

Oscillation of Second Order Nonlinear Difference Equation with Nonlinear Damping Term

4.1 Introduction

Analyzing the conditions for oscillation nonlinear difference equations of second order is necessary to understand discrete systems' dynamics. Researchers Sh. R. Elzeiny and Z. A. Elmaned (2018) have used Riccati techniques to create new methods that enhance the conventional framework. These methods look at the sufficient and necessary conditions for oscillation. Their research, as reported in (Sh. R. Elzeiny et al., 2018), uses a rigorous methodology that takes contradictions and limit superior into account to show the complexity of oscillatory solutions.

By specifying exact sequential behaviors and summation requirements, the authors show that the current standard results in the field cannot take into account the complexities of these equations. This nuanced perspective advances a deeper comprehension of the oscillatory nature of solutions, which enhances the theoretical framework and provides useful applications in a number of scientific domains. As a result, the development of Riccati methods aids in the future development of the analysis of nonlinear difference equations while also providing clarification on oscillation events.

In this chapter, we will study of the sufficient conditions for the oscillation of the solutions of the second order non-linear difference equation of the form

$$\Delta(c_m \Delta x_m) + \theta_m\, \varphi(x_m, \Delta x_m) + q_{m+1} f(x_{m+1}) = 0 \ , m \in \mathbb{N} \qquad (E)$$

where Δ denotes the forward difference operator $\Delta x_m = x_{m+1} - x_m$ for any sequence $\{x_m\}$ of real numbers, p_n , q_n are real sequences, the function $f: \mathbb{R} \to \mathbb{R}$ is continuous, the function φ is defined on $\mathbb{R} \times \mathbb{R}$ and p_n , q_n , f and φ satisfy the following conditions:

(C_1) The sequences r_m , θ_m , q_m are non-negative for each $m \in \mathbb{N}$.

(C_2) For every $y \neq 0$ and $x \in \mathbb{R}$ there exists a constant $L > 0$such that

$0 < \varphi(x, y)y \leq Ly^2$.

(C_3) f is a nondecreasing function such that $x\, f(x) > 0$, for all $x \neq 0$.

A nontrivial solution of *(E)* is a real sequence $\{x_m\}$ that is not exactly equal to zero for $m \geq 0$ and satisfies equation *(E).*

4.2 The oscillatory solutions for equation *(E)*

This section outlines scenarios that lead to oscillation in the equation, accompanied by examples that clarify and support these observations. The following lemmas will be essential for this chapter.

Lemma 4.2.1 If $x_m \geq 0$, $\Delta x_m > 0$ and $\Delta^2 x_m \leq 0$ for $m \geq N$, then $x_{m+1} \geq \frac{m}{2} \Delta x_m$ for all $m \geq 2N$.

Proof. We have

$$\Delta^2 x_m = \Delta(\Delta x_m) = \Delta(x_{m+1} - x_m) = \Delta x_{m+1} - \Delta x_m$$

$$\Delta x_{m+1} \leq \Delta x_m$$

Writing the sequence x_m as a sum

$$x_m = x_1 + \sum_{n=1}^{m-1} \Delta x_n$$

where

$x_1 \geq 0$ and $\Delta x_n > 0\ \forall\, n$

Since $\Delta^2 x_m \leq 0$, then Δx_m decreases but remains positive [$\Delta x_m > 0$].

So we can estimate Δx_m for n as follows :

Since Δx_m gradually decreases, then

$$\Delta x_n \geq \Delta x_m \quad , n \geq m$$

Therefore

$$x_m \geq \sum_{n=1}^{m} \Delta x_m = m\, \Delta x_m$$

We have,

$$x_{m+1} = x_m + \Delta x_m$$

By using $x_m \geq m\, \Delta x_m$

$$x_{m+1} \geq (\mathrm{m}+1)\Delta x_m$$

Compare (m+1) with $\frac{\mathrm{m}}{2}$ for all $m \geq 2N$

Dividing both sides by $\Delta x_m > 0$, we obtain

$$\frac{m}{2}+1>0 \quad \forall\, \mathrm{m}>2\mathrm{N}$$

Since $\mathrm{m}+1 \geq \frac{m}{2}$ for all $\mathrm{m} \geq 2N$, then

$x_{m+1} \geq \frac{m}{2}\Delta x_m$ for all $m \geq 2N$.

Lemma 4.2.2 Let conditions (C_1)-(C_3) hold. Assume that

(C_4) $c_m - L\theta_m > 0 \;\; \forall\, m \geq N_0 \in \mathbb{N}$

and

(C_5) $\sum_{m=N_0}^{\infty} \frac{1}{c_m}\left[\prod_{s=N_0}^{m-1}\left(1-\frac{L\theta_s}{c_s}\right)\right] = \infty.$

Then if $\{x_m\}$ is a non oscillator solution of equation *(E),* we have $x_m\,\Delta x_m > 0$ for all sufficiently large m.

Proof. Let $\{x_m\}$ be a non-oscillator solution of equation (*E*) and assume that $x_m > 0$ for $m \geq N_0 \in \mathbb{N}$. Suppose $\{\Delta x_m\}$ is oscillatory, then there exists an integer $N_1 \geq N_0 > 0$ such that $\Delta x_{N_1} < 0$ or $\Delta x_{N_1} = 0$.

First, we consider $\Delta x_{N_1} < 0$ **.** Now from equation (*E*) we have

$$\Delta(c_{N_1}\Delta x_{N_1})\Delta x_{N_1} + \theta_{N_1}\,\varphi\big(x_{N_1},\Delta x_{N_1}\big)\,\Delta x_{N_1} = -\,q_{N_1+1}f(x_{N_1+1})\Delta x_{N_1} > 0$$

$$\Delta(c_{N_1}\Delta x_{N_1})\Delta x_{N_1} + \theta_{N_1}\,\varphi\big(x_{N_1},\Delta x_{N_1}\big)\,\Delta x_{N_1} > 0$$

$$\Delta(c_{N_1}\Delta x_{N_1})\Delta x_{N_1} > -\theta_{N_1}\,\varphi\big(x_{N_1},\Delta x_{N_1}\big)\,\Delta x_{N_1} \tag{4.1}$$

But

$$\begin{aligned}\Delta(c_{N_1}\Delta x_{N_1})\Delta x_{N_1} &= \big(c_{N_1+1}\Delta x_{N_1+1} - c_{N_1}\Delta x_{N_1}\big)\Delta x_{N_1}\\ &= c_{N_1+1}\Delta x_{N_1+1}\Delta x_{N_1} - c_{N_1}\big(\Delta x_{N_1}\big)^2 \qquad (4.2)\end{aligned}$$

Using (C_2) we get

$$0 < \varphi\big(x_{N_1},\Delta x_{N_1}\big)\Delta x_{N_1} \leq L\big(\Delta x_{N_1}\big)^2$$

$$-\theta_{N_1}\,\varphi\big(x_{N_1},\Delta x_{N_1}\big)\,\Delta x_{N_1} \geq -\theta_{N_1}L\big(\Delta x_{N_1}\big)^2 \tag{4.3}$$

So, from (4.1) we have

$$c_{N_1+1}\Delta x_{N_1+1}\Delta x_{N_1} - c_{N_1}\big(\Delta x_{N_1}\big)^2 \geq -\theta_{N_1}L\big(\Delta x_{N_1}\big)^2,$$

or

$$c_{N_1+1}\Delta x_{N_1+1}\Delta x_{N_1} \geq c_{N_1}\big(\Delta x_{N_1}\big)^2 - \theta_{N_1}L\big(\Delta x_{N_1}\big)^2$$

$$= \left(\Delta x_{N_1}\right)^2 \left[c_{N_1} - \theta_{N_1} L\right]$$

$$c_{N_1+1}\Delta x_{N_1+1} \leq \Delta x_{N_1}\left[c_{N_1} - \theta_{N_1} L\right].$$

Since $c_{N_1} - \theta_{N_1} > 0,\ \Delta x_{N_1} < 0$ then

$$\Delta x_{N_1+1} \leq \frac{1}{c_{N_1+1}} \Delta x_{N_1}\left[c_{N_1} - \theta_{N_1} L\right] < 0,$$

which leads us to $\Delta x_{N_1+1} < 0 \ \Rightarrow \Delta x_{N_1} < 0$ for each $m > N_1 \in \mathbb{N}$.

Now, study the case $\Delta x_{N_1} = 0$, then from eq. (E) we obtain

$$\Delta x_{N_1+1} < 0 \ \Rightarrow \Delta x_{N_1} < 0 \ \forall \ m \geq N_1.$$

Observe that, in both cases, we have $\Delta x_{N_1} < 0 \ \forall \ m \geq N_1$ It, however, runs counter to the notion that $\{x_m\}$ oscillates. Δx_m is a fixed sign in that case.

Let $\Delta x_m \ < 0\ \forall \ m \geq N_1$. From equation ($E$) and ($C_2$) we can see

$$\Delta(c_m \Delta x_m) = -\theta_m\, \varphi(x_m, \Delta x_m) - q_{m+1} f(x_{m+1}) \ \ , m \in \mathbb{N}$$

$$\Delta(c_m \Delta x_m) \leq -\theta_m\, \varphi(x_m, \Delta x_m),$$

And by using definition of the operator Δ and (C_2) we obtain

$$c_{m+1}\Delta x_{m+1}\Delta x_m \geq c_m(\Delta x_m)^2 + L\,\theta_m(\Delta x_m)^2$$

$$c_{m+1}\Delta x_{m+1} \leq c_m\Delta x_m + L\,\theta_m + \Delta x_m$$

$$c_{m+1}\Delta x_{m+1} - c_m\Delta x_m \leq L\,\theta_m\Delta x_m$$

$$\Delta w_m \geq -L$$

$$\Delta w_m + L\frac{\theta_m}{c_m} w_m \geq 0,$$

where $w_m = -c_m\Delta x_m$.

$$w_m - w_{N_0} \geq -L\frac{\theta_m}{c_m} w_{N_0}. \qquad (4.4)$$

From (4.4) we obtain

$$w_m \geq \left(1 - L\frac{\theta_m}{c_m}\right) w_{N_0}$$

$$w_m \geq w_{N_0} \prod_{S=N_0}^{m-1}\left(1 - L\frac{\theta_s}{c_s}\right)$$

$$-\Delta x_m \ \geq w_{N_0}\frac{1}{c_m}\prod_{S=N_0}^{m-1}\left(1 - L\frac{\theta_s}{c_s}\right)$$

$$\Delta x_m \leq -w_{N_0}\frac{1}{c_m}\prod_{S=N_0}^{m-1}\left(1 - L\frac{\theta_s}{c_s}\right). \qquad (4.5)$$

Summing inequality (4.5) from N_0 to $m-1$ we get

$$x_m \leq x_{N_0} - w_{N_0} \frac{1}{c_m} \prod_{S=N_0}^{m-1} \left(1 - L\frac{\theta_s}{c_s}\right)$$

Given (C_5), we have $x_m \to -\infty$ as $m \to \infty$, which is contradiction.

Theorem 4.2.1 Assume these conditions (C_1) –(C_5) and

(C_6) $\int^{-\infty} \frac{dv}{f(v)} < \infty \quad and \quad \int^{\infty} \frac{dv}{f(v)} < \infty,$

are satisfied. Assume that a positive sequence exists $\{\rho_m\}$ such that

(C_7) $\Delta\rho_m \geq 0$ and $\Delta(c_m\Delta\rho_m) \leq 0 \quad \forall\, m \geq n_0 \geq 0.$

If

(C_8) $\sum_{n_0}^{\infty} \rho_m q_{m+1} = \infty$.

Then every solution of equation *(E)* is oscillatory.

Proof. Let $\{x_m\}$ be a solution of (E) that does not oscillate, this must finally have a constant sign as a result. Considering Lemma 4.2.2, it is reasonable to assume the existence of $N \geq 0$ is an integer such that for any $m \geq N$, $x_m > 0$ and $\Delta x_m > 0$.

Define

$$w_m = \frac{\rho_m u_m}{f(x_m)}$$

where $u_m = c_m \Delta x_m$.

Then for $m \geq N$

$$\Delta w_m = w_{m+1} - w_m = \frac{\rho_{m+1}u_{m+1}}{f(x_{m+1})} - \frac{\rho_m u_m}{f(x_m)}$$

$$= \frac{f(x_m)\rho_{m+1}u_{m+1} - \rho_m u_m f(x_{m+1})}{f(x_m)f(x_{m+1})}$$

By adding and subtracting $\rho_m u_{m+1} f(x_m)$ in the numerator we get

$$\Delta w_m = \frac{f(x_m)\rho_{m+1}u_{m+1} - \rho_m u_m f(x_{m+1}) + \rho_m u_{m+1} f(x_m) - \rho_m u_{m+1} f(x_m)}{f(x_m)f(x_{m+1})}$$

$$= \frac{f(x_m)u_{m+1}(\rho_{m+1} - \rho_m) + \rho_m\big(u_{m+1}f(x_m) - u_m f(x_{m+1})\big)}{f(x_m)f(x_{m+1})}$$

$$= \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} + \frac{\rho_m(u_{m+1}f(x_m) - u_m f(x_{m+1}))}{f(x_m)f(x_{m+1})}$$

Adding and subtracting $u_m f(x_m)$ we have

$$\Delta w_m = \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} + \frac{\rho_m\big(u_{m+1}f(x_m) - u_m f(x_m) + u_m f(x_m) - u_m f(x_{m+1})\big)}{f(x_m)f(x_{m+1})}$$

$$= \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} + \frac{\rho_m[f(x_m)(u_{m+1} - u_m) - u_m(f(x_{m+1}) - f(x_m))]}{f(x_m)f(x_{m+1})}$$

$$\Delta w_m = \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} + \frac{\rho_m[f(x_m)\Delta u_m - u_m\Delta f(x_m)]}{f(x_m)f(x_{m+1})}. \qquad (4.6)$$

Now from Equation *(E)* we find that

$$\Delta u_m = -\theta_m\, \varphi(x_m, \Delta x_m) - q_{m+1} f(x_{m+1})$$

By substituting in (4.6) We have

$$\Delta w_m = \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} + \frac{\rho_m\big[f(x_m)\big(-\theta\, \varphi(x_m, \Delta x_m) - q_{m+1} f(x_{m+1})\big) - u_m \Delta f(x_m)\big]}{f(x_m)f(x_{m+1})}$$

$$\Delta w_m = \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} - \frac{\rho_m\theta_m\, \varphi(x_m, \Delta x_m)}{f(x_{m+1})} - \rho_m q_{m+1} - \frac{\rho_m u_m \Delta f(x_m)}{f(x_{m+1})}$$

Since

$\Delta\rho_m \geq 0,\ \Delta(c_m\Delta\rho_m) \leq 0\ \ \forall\, m \geq n_0\ and\ \ \varphi(x_m, \Delta x_m) > 0$, then

$$\frac{\theta_m\, \varphi(x_m, \Delta x_m)}{f(x_{m+1})} > 0$$

$and\ u_{m+1} \leq u_m,$

so,

$$\Delta w_m \leq -\rho_m q_{m+1} + \frac{u_m\Delta\rho_m}{f(x_{m+1})}, \qquad m \geq n_0.$$

Since $\Delta(c_m\Delta\rho_m) \leq 0$ we can deduce $c_m\Delta\rho_m$ is do not increase sequence for $\forall\, m \geq n_0$ therefore, $c_m\Delta\rho_m \leq c_{n_0}\Delta\rho_{n_0}$

$$\Delta w_m \leq -\rho_m q_{m+1} + c_{n_0}\Delta\rho_{n_0} \frac{\Delta x_m}{f(x_{m+1})}, \qquad m \geq n_0. \qquad (4.7)$$

Now, since f is a non-decreasing function then $f(x) \leq f(y_{n+1})$ for each $y \in [x_m, x_{m+1}]$,

$$\frac{1}{f(y)} \geq \frac{1}{f(x_{m+1})},$$

Integrating on $[x_m, x_{m+1}]$, we obtain

$$\int_{x_m}^{x_{m+1}} \frac{dy}{f(y)} \geq \int_{x_m}^{x_{m+1}} \frac{dy}{f(x_{m+1})}$$

$$= \frac{x_{m+1}-x_m}{f(x_{m+1})} = \frac{\Delta x_m}{f(x_{m+1})},$$

$$\frac{\Delta x_m}{f(x_{m+1})} \leq \int_{x_m}^{x_{m+1}} \frac{dy}{f(y)} \qquad (4.8)$$

Using (4.8) in (4.7) and by summing (4.7) from n_0 to $m-1$ we obtain,

$$\sum_{s=n_0}^{m-1} \Delta w_s < -\sum_{s=n_0}^{m-1} \rho_s q_{s+1} + c_{n_0}\Delta\rho_{n_0} \sum_{s=n_0}^{m-1} \int_{x_m}^{x_{m+1}} \frac{dy}{f(y)} \qquad m \geq n_0.$$

Now, $\sum_{s=n_0}^{m-1} \Delta w_s = w_{m+1-1} - w_{n_0} = w_m - w_{n_0},$

$$c_{n_0}\Delta\rho_{n_0} \sum_{s=n_0}^{m-1} \int_{x_m}^{x_{m+1}} \frac{dy}{f(y)} = c_{n_0}\Delta\rho_{n_0} \int_{x_{n_0}}^{x_m} \frac{dy}{f(y)}.$$

Therefore,

$$w_m - w_{n_0} < -\sum_{s=n_0}^{m-1} \rho_s q_{s+1} + c_{n_0}\Delta\rho_{n_0} \int_{x_{n_0}}^{x_m} \frac{dy}{f(y)}.$$

$$\sum_{s=n_0}^{m-1} \rho_s q_{s+1} < w_{n_0} - w_m + c_{n_0}\Delta\rho_{n_0} \int_{x_{n_0}}^{x_m} \frac{dy}{f(y)}.$$

Let's look at the right side when $m \to \infty$

a) $w_m > 0$ then $w_{n_0} - w_m < w_{n_0}$.

b) $c_{n_0}\Delta\rho_{n_0}$ is a constant.

c) Since $x_m > 0$ and $\Delta x_m > 0$, then x_m tends either to a finite limit L > 0 or to ∞.

d) From (C_6) the integral $\int_{x_{n_0}}^{x_m} \frac{dy}{f(y)}$ is bounded from above by the convergent integral $\int_{x_{n_0}}^{\infty} \frac{dy}{f(y)}$.

So, the entire right-hand side is bounded from above by a constant value when $m \to \infty$

$$\sum_{s=n_0}^{n-1} \rho_s q_{s+1} < w_{n_0} - w_m + c_{n_0}\Delta\rho_{n_0} \int_{x_{n_0}}^{\infty} \frac{dy}{f(y)} = C. \qquad (4.9)$$

$$\sum_{s=n_0}^{n} \rho_s q_{s+1} < C,$$

Taking the limit as $m \to \infty$ in (4.9), the left side becomes $\sum_{s=n_0}^{m-1} \rho_s q_{s+1}$ which diverges to ∞ by condition (C_8). For the right side, $w_m > 0, -w_m \leq 0$. By condition (C_6), the integral $\int_{x_{n_0}}^{x_m} \frac{dy}{f(y)}$ is bounded above by the convergent integral $\int_{x_{n_0}}^{\infty} \frac{dy}{f(y)}$. Thus, the right side $w_{n_0} - w_m + c_{n_0} \Delta\rho_{n_0} \int_{x_{n_0}}^{\infty.} \frac{dy}{f(y)}$ is bounded above by a constant value as $m \to \infty$. This leads to the contradiction.

Remark 4.2.1 In theorem 4.2.1, if $c_m = 1$, then it reduces to Theorem 1 of (Thandapani & Pandian, 1995).

Example 4.2.1 Examine the difference equation.

$$\Delta\big((m+3)\Delta x_m\big) + \frac{1}{m+3}\Delta x_m + (m+3)(4m^2+10m+5)(x_{m+1} + x_{m+1}^3) = 0,$$

$m \geq 1.$

where

$c_m = (m+3) > 0,$

$\theta_m = \frac{1}{m+3} > 0,$

$q_{m+1} = (m+3)(4m^2+10m+5) > 0$

$\varphi(x_m, \Delta x_m) = \Delta x_m \to \varphi(x_m, \Delta x_m)\Delta x_m = (\Delta x_m)^2 \leq (\Delta x_m)^2, L = 1,$

$f(x) = x + x^3 \;\; \to \;\; x f(x) = x^2 + x^4 \; > 0$

Condition (C_4) is hold because

$$c_m - L\theta_m = m + 3 - \frac{1}{m+3} > 0, \quad \forall\, m \geq 1,$$

Therefore, for a positive constant L, the condition is satisfied for all m ≥ 1where $L = 1$

.

and(C_2) is also hold because

$$\sum_{n=N_0}^{\infty} \frac{1}{c_m}\left[\prod_{s=N_0}^{m-1}\left(1 - \frac{L\theta_s}{c_s}\right)\right] = \sum_{n=N_0}^{\infty} \frac{1}{(m+3)}\left[\prod_{s=N_0}^{m-1}\left(1 - \frac{\frac{1}{s+3}}{s+3}\right)\right].$$

Thus, the product becomes

$$\prod_{s=1}^{m-1}\left(1 - \frac{1}{(m+3)^2}\right) \cong \frac{3}{4}\frac{(m+3)}{(m+2)}.$$

Using know properties of logarithmic approximations, if L =1, then the infinite sum

$$\sum_{m=1}^{\infty} \frac{3}{4(m+2)} = \infty,$$

$$\int_{-\infty}^{\infty} \frac{dv}{f(v)} = \int_{-\infty}^{\infty} \frac{dv}{v+v^3} = \infty,$$

Let $\rho_m = 1 \ \rightarrow \Delta\rho_m = 0, \Delta(c_m \Delta\rho_m) = 0$

and (C_8) is also hold because

$$\sum_{m}^{\infty} \rho_m q_{m+1} = \sum_{m=1}^{\infty} (m+3)(4m^2 + 10m + 5)$$

$$= \sum_{m}^{\infty} (m+3)(4m^2 + 10m + 5) = \infty$$

Hence,

$$\sum_{m}^{\infty} \rho_m q_{m+1} = \infty$$

Then, according to Theorem 4.2.1, the difference equation's solutions are oscillatory.

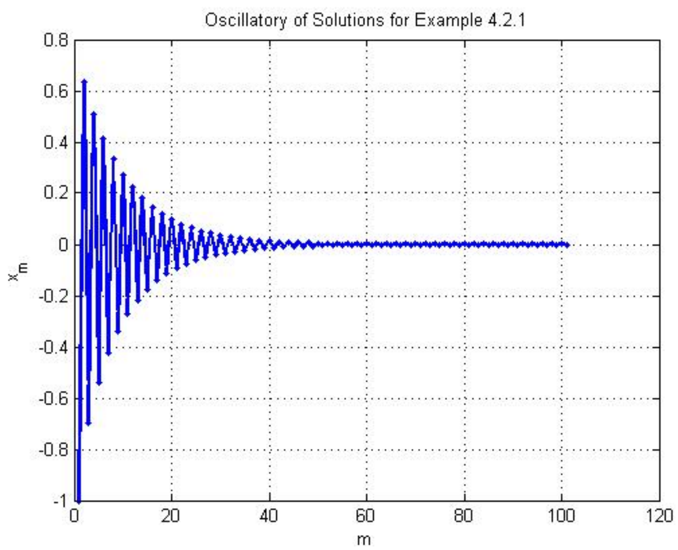

Fig 4.1: Oscillatory of solutions for example 4.2.1

In the sequel we need the following lemma of which is due to Thandapani (1995).

Lemma 4.2.3 (Thandapani,1995)

If $x_m \geq 0, \Delta(c_m \Delta x_m) \leq 0$ and $\Delta x_m > 0$ for $m \geq n_0 \geq 1$, then $x_{m+1} \geq \sum_{s=n_0}^{m} \frac{1}{c_s}(c_m \Delta x_m)$.

Theorem 4.2.2 Assume conditions $(C_1) - (C_5)$ satisfied. If

$$(C_9) \quad \int^{\pm a} \frac{dv}{f(v)} < \infty \quad \forall a > 0,$$

(C_{10}) $\quad f$ does not decrease and $f(xy) \geq \gamma f(x) f(y)$ and $-f(-xy) \geq \gamma f(x) f(y)$ for some constant $\gamma > 0, x \; and \; y \; \in \mathbb{R}\backslash\{0\}$

and

$$(C_{11}) \sum_{s}^{\infty} f\big(\beta(m)\big) q_{m+1} = \infty, \quad \beta(m) = \sum_{s=n_0}^{m} \frac{1}{c_s}$$

Then every solution of eq. *(E)* is oscillatory.

Proof. Let$\{x_m\}$ be a non-oscillatory solution of *(E)*, i.e there exists an integer $n_0 \geq 0$ such that $x_m > 0, \Delta x_m > 0 \;\; \forall m \geq n_0$.

According to Lemma 4.2.3 if $\Delta(c_m \Delta x_m) \leq 0$ and $\Delta x_m > 0$ for $m \geq n_0 \geq 1$, then $x_{m+1} \geq \sum_{s=n_0}^{m} \frac{1}{c_s}(c_m \Delta x_m)$.

Substitute in equation *(E)* and by using $(C_1) - (C_2)$ we have

$$\Delta(c_m \Delta x_m) + q_{m+1} f(x_{m+1}) \leq 0.$$

From Lemma 4.2.3 in the function f we can see

$$f(x_{m+1}) \geq f(\beta(m) c_m \Delta x_m); \quad \beta(m) = \sum_{s=n_0}^{m} \frac{1}{c_s}.$$

The condition (C_{10}) stats:

$$f(xy) \geq \gamma f(x) f(y) \text{ for } x, y > 0.$$

Let $x = \beta(m), \quad y = c_m \Delta x_m,$ we obtain

$$f(\beta(m) c_m \Delta x_m) \geq \gamma f\big(\beta(m)\big) f(c_m \Delta x_m).$$

Then

$$\Delta(c_m \Delta x_m) + \gamma q_{m+1} f\big(\beta(m)\big) f(c_m \Delta x_m) \leq 0.$$

Divide both sides by $f(c_m \Delta x_m)$ we get

$$\frac{\Delta(c_m \Delta x_m)}{f(c_m \Delta x_m)} \leq -\gamma q_{m+1} f\big(\beta(m)\big)$$

Note that for $c_m\Delta x_m \geq y \geq c_{m+1}\Delta x_{m+1}$ we have

$$\frac{1}{f(y)} \geq \frac{1}{f(c_m\Delta x_m)}$$

$$\int_{c_{m+1}\Delta x_{m+1}}^{c_m\Delta x_m} \frac{dy}{f(y)} \geq \int_{c_{m+1}\Delta x_{m+1}}^{c_m\Delta x_m} \frac{dy}{f(c_m\Delta x_m)} = \frac{c_m\Delta x_m - c_{m+1}\Delta x_{m+1}}{f(c_m\Delta x_m)}$$

$$= \frac{-[\Delta(c_m\Delta x_m\,)]}{f(c_m\Delta x_m)},$$

so,

$$\int_{c_{m+1}\Delta x_{m+1}}^{c_m\Delta x_m} \frac{dy}{f(y)} \geq \frac{-[\Delta(c_m\Delta x_m\,)]}{f(c_m\Delta x_m)}$$

$$-\int_{c_{m+1}\Delta x_{m+1}}^{c_m\Delta x_m} \frac{dy}{f(y)} \leq \frac{[\Delta(c_m\Delta x_m\,)]}{f(c_m\Delta x_m)}.$$

Then,

$$\frac{[\Delta(c_m\Delta x_m\,)]}{f(c_m\Delta x_m)} \geq -\int_{c_{m+1}\Delta x_{m+1}}^{c_m\Delta x_m} \frac{dy}{f(y)} \tag{4.10}$$

We summation the inequality (4.10) from n_0 to m we get,

$$\gamma \sum_{s=n_0}^{m} f\big(\beta(m)\big)\, q_{m+1} \leq \int_{c_{m+1}\Delta x_{m+1}}^{c_{n_0}\Delta x_{n_0}} \frac{dy}{f(y)}$$

According to condition (C_9), the right-handed integral, is bounded, However, the sum is on the left (C_{11}) side by condition unbounded. This is a contradiction.

Example 4.2.2 Consider the difference equation.

$$\Delta\big((m+2)\Delta x_m\big) + \frac{1}{m+2}\Delta x_m + \frac{(4m^2+8m+5)}{(m+2)^{\frac{5}{3}}} x_{m+1}^{\frac{1}{3}} = 0, m \geq 1.$$

Where

$c_m = (m+2) > 0,$

$\theta_m = \frac{1}{m+2} > 0,$

$$q_{m+1} = \frac{(4m^2+8m+5)}{(m+2)^{\frac{5}{3}}} > 0$$

Because the fraction is positive for all $m \in \mathbb{N}$.

$\varphi(x_m, \Delta x_m) = \Delta x_m \to \varphi(x_m, \Delta x_m)\Delta x_m = (\Delta x_m)^2 \leq (\Delta x_m)^2, L = 1,$

$f(x) = x^{\frac{1}{3}} \rightarrow \ xf(x) = x^{\frac{4}{3}} > 0$

Condition (O_1) is met because

$$c_m - L\theta_m = m + 2 - \frac{1}{m+2} > 0, \quad \forall\, m \geq 1,$$

Therefore, for a positive constant L, the condition is satisfied all m ≥ 1where $L = 1$.

We not that,

$$\int^{a} \frac{dv}{f(v)} = \int^{a} \frac{dv}{v^{1/3}} = \frac{3}{2} a^{\frac{2}{3}}$$

The integral is finite. Similarly, it can be show that $\int^{-a} \frac{dv}{f(v)} < \infty$.

$f(xy) = (xy)^{1/3} = x^{1/3} y^{1/3} = f(x)f(y), \ \ x, y \ > 0$ and $\gamma = 1$

and

$-f(-xy) = -(-xy)^{1/3} = x^{1/3} y^{1/3} = f(x)f(y), x, y \ < 0 \ \ and\ \gamma = 1,$

$$\sum_{s}^{\infty} f\big(\beta(m)\big) q_{m+1} = \infty, \quad \beta(m) = \sum_{s=n_0}^{m} \frac{1}{c_s},$$

$$\sum_{m=1}^{\infty} \frac{(4m^2 + 8m + 5)}{(m+2)^{\frac{5}{3}}} \sum_{s=1}^{m} \left(\frac{1}{s+2}\right)^3 = \infty,$$

Thus, it follows from theorem 4.2.2 that the equation is oscillatory.

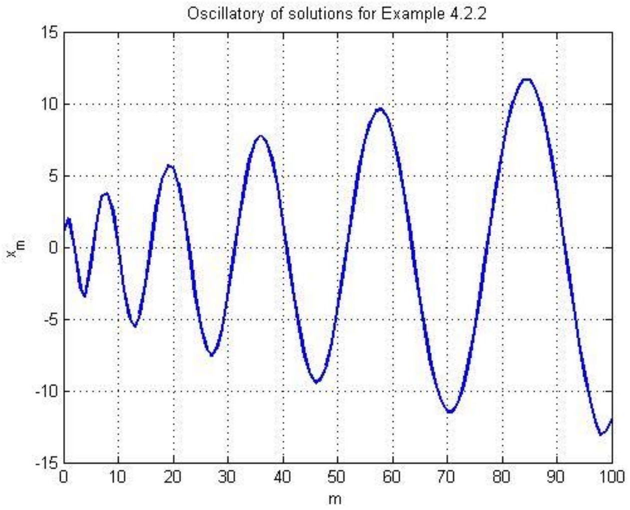

Fig 4.2: Oscillatory of solutions for example 4.2.2

Theorem 4.2.3 In addition to (C_4),(C_5), (C_9) assume that $\sigma > 0$ and positive non decreasing sequence $\{\rho_m\}_{m \geq N}$ such that

(C_{12}) $\Delta(\Delta \rho_m) \leq 0$ and $\theta_m \geq -\frac{\sigma}{\rho_m}$ for all $m \geq N > 0$.

If

$$(C_{13})\sum \rho_m\, q_{m+1} = \infty.$$

Then every solution of *(E)* are oscillatory.

Proof. Assume that the non-oscillatory solution of *(E)* is $\{x_m\}$. We may suppose that there is an integer $N \geq 0$ such that that $x_m > 0$ and $\Delta x_m > 0$ for all $m \geq N$ without loss of generality in light of Lemma 4.2.2 .

Describe

$$w_m = \frac{\rho_m u_m}{f(x_m)}$$

Where $u_m = c_m \Delta x_m$

Then for $m \geq N$

$$\Delta w_m = \Delta\left(\frac{\rho_m u_m}{f(x_m)}\right) = \frac{f(x_m)\Delta(\rho_m u_m) - \rho_m u_m \Delta f(x_m)}{f(x_m) f(x_{m+1})}$$

$$= \frac{f(x_m)\Delta(\rho_m u_m)}{f(x_m) f(x_{m+1})} - \frac{\rho_m u_m \Delta f(x_m)}{f(x_m) f(x_{m+1})}$$

$$= \frac{\Delta(\rho_m u_m)}{f(x_{m+1})} - \frac{\rho_m u_m \Delta f(x_m)}{f(x_m) f(x_{m+1})}$$

We have

$$\frac{\Delta(\rho_m u_m)}{f(x_{m+1})} = \frac{u_{m+1}\Delta\rho_m + \rho_m \Delta(u_m)}{f(x_{m+1})} = \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} + \frac{\rho_m \Delta(u_m)}{f(x_{m+1})}$$

From equation *(E)*,

$$\Delta u_m = -\theta_m\, \varphi(x_m, \Delta x_m) - q_{m+1} f(x_{m+1})$$

$$\Delta w_m = \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} + \frac{\rho_m \Delta(u_m)}{f(x_{m+1})} - \frac{\rho_m u_m \Delta f(x_m)}{f(x_m) f(x_{m+1})}$$

$$= \frac{\rho_m}{f(x_{m+1})}[-\theta_m\, \varphi(x_m, \Delta x_m) - q_{m+1} f(x_{m+1})] + \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} - \frac{\rho_m u_m \Delta f(x_m)}{f(x_m) f(x_{m+1})}$$

$$= -\rho_m q_{m+1} - \frac{\rho_m\, \theta_m\, \varphi(x_m, \Delta x_m)}{f(x_{m+1})} + \frac{u_{m+1}\Delta\rho_m}{f(x_{m+1})} - \frac{\rho_m u_m \Delta f(x_m)}{f(x_m) f(x_{m+1})} \tag{4.11}$$

Now using the condition $(C_{13})$and $\Delta x_{m+1} \leq \Delta x_m$ in (4.11), we obtain

$$\Delta w_m \leq -\rho_m q_{m+1} + \frac{u_m \Delta\rho_m}{f(x_{m+1})} \qquad \text{for } m \geq N$$

Since $\Delta\rho_m$ is non increasing for $m \geq N$, we have

$$\Delta w_m < -\rho_m q_{m+1} + \Delta\rho_N \frac{u_m}{f(x_{m+1})} \;, m \geq N \tag{4.12}$$

Now for $x_m \leq y \leq x_{m+1}$, we have

$$\frac{1}{f(y)} \geq \frac{1}{f(x_{m+1})}$$

and it follows that,

$$\int_{x_m}^{x_{m+1}} \frac{dy}{f(y)} \geq \frac{\Delta x_m}{f(x_{m+1})}$$

Using the above inequality in (4.12) and summing the resulting inequality from *N* to *m* leads to

$$w_{m+1} - w_m < -\rho_m q_{m+1} + \Delta\rho_N \int_{x_m}^{x_{m+1}} \frac{dy}{f(y)}$$

$$\sum_{s=N}^{m} \rho_s q_{s+1} < w_N - w_{m+1} + \Delta\rho_N \int_{x_m}^{x_{m+1}} \frac{dy}{f(y)}$$

In view of (C_{13}) and $w_m > 0$, $m \geq N$, the above inequality gives

$$\sum_{s=N}^{m} \rho_s q_{s+1} < \infty$$

Which contradicts (C_{13}).

Example 4.2.3 Consider the difference equation

$$\Delta\big((m+1)\Delta x_m\big) + \frac{1}{m+1}\Delta x_m + \frac{1}{\sqrt{m+1}} x_{m+1}^{\frac{1}{3}} = 0, m \geq 1.$$

Condition $(C_4)$is hold because

$$c_m - L\theta_m = m + 1 - \frac{1}{m+1} > 0, \quad \forall\, m \geq 1,$$

Therefore, for a positive constant L, the condition is satisfied all m ≥ 1where $L = 1$.

We not that, (C_9) is hold because

$$\int^{a} \frac{dv}{f(v)} = \int^{a} \frac{dv}{v^{1/3}} = \frac{3}{2} a^{\frac{3}{2}}$$

The integral is finite. Similarly, it can be show that $\int^{-a} \frac{dv}{f(v)} < \infty$.

We take

$\rho_m = \sqrt{m+1}$

$(\Delta\rho_m) = \rho_{m+1} - \rho_m = \sqrt{m+2} - \sqrt{m+1}$

$\Delta(\Delta\rho_m) = \left(\sqrt{m+3} - \sqrt{m+2}\right) - \left(\sqrt{m+2} - \sqrt{m+1}\right)$

$\qquad\qquad = \sqrt{m+3} - 2\sqrt{m+2} + \sqrt{m+1}$

Using Taylor's approximation, it can be proven that $\Delta(\Delta\rho_m) < 0$.
Therefor, the condition $\Delta(\Delta\rho_m) < 0$ is satisfied.

$$\sum \rho_m \, q_{m+1} = \sqrt{m+1}\,\frac{1}{\sqrt{m+1}} = \infty$$

Thus, from theorem 4.2.3 it follows that the equation is oscillatory.

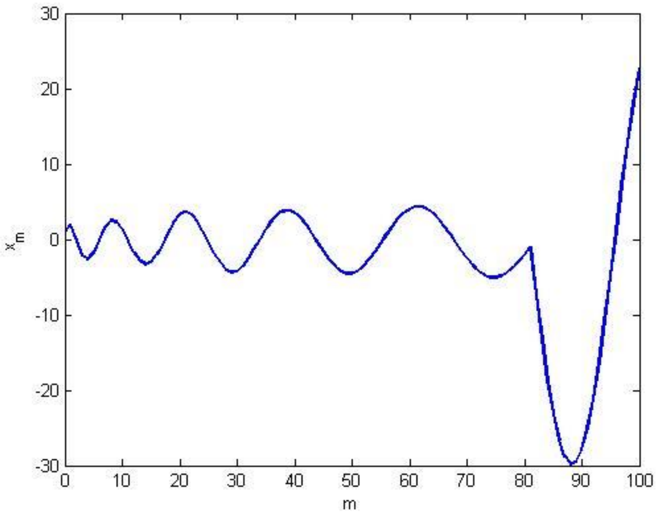

Fig 4.3: Oscillatory of solutions for example 4.2.3

Theorem 4.2.4 In addition to $(C_1) - (C_5)$ and (C_{14}) assume that if there α is the ratio of odd positive integers holds for $\alpha = 1$, such that

$\frac{f(x)}{x^{\alpha}} \geq L > 0$ for all $x \neq 0$, for some constant L.

suppose there exists a positive sequence such that

$$(C_{15}) \sum_{s=N}^{\infty} \rho_s \left[q_{s+1} - \left(\frac{\Delta\rho_s}{2\rho_s}\right)^2 \right] = \infty$$

Then every solution of equation *(E)* is oscillatory.

Proof. Let $\{x_m\}$ be a solution to *(E)* that does not oscillate, and based on Lemma 4.2.2, by generality preserving, assume there exists an integer $N \geq 0$ such that $x_m > 0$ and $\Delta x_m > 0 \quad \forall\, m \geq N > 1$.

Define

$$w_m = \frac{\rho_m c_m \Delta x_m}{x_m}$$

Then, we get

$$\Delta w_m = \Delta\left(\frac{\rho_m c_m \Delta x_m}{x_m}\right)$$

$$\Delta w_m = \Delta\left(\frac{\rho_m c_m \Delta x_m}{x_m}\right) = \frac{x_m\ \Delta(\rho_m c_m \Delta x_m) - \rho_m \Delta x_m c_m \Delta x_m}{x_m x_{m+1}}$$

$$= \frac{x_m\ \Delta(\rho_m c_m \Delta x_m)}{x_m x_{m+1}} - \frac{\rho_m \Delta x_m c_m \Delta x_m}{x_m x_{m+1}}$$

$$= \frac{\Delta(\rho_m c_m \Delta x_m)}{x_{m+1}} - \frac{\rho_m \Delta x_m c_m \Delta x_m}{x_m x_{m+1}}$$

We have

$$\frac{\Delta(\rho_m c_m \Delta x_m)}{x_{m+1}} = \frac{\Delta x_{m+1} c_{m+1} \Delta\rho_m + \rho_m \Delta(c_m \Delta x_m)}{x_{m+1}} = \frac{\Delta x_{m+1} c_{m+1} \Delta\rho_m}{x_{m+1}} + \frac{\rho_m \Delta(c_m \Delta x_m)}{x_{m+1}}$$

From equation *(E)*,

$$\Delta(c_m \Delta x_m) + q_{m+1} f(x_{m+1}) = -\,\theta_m\ \varphi(x_m, \Delta x_m)$$

$$\rho_m \Delta(c_m \Delta x_m) + \rho_m q_{m+1} f(x_{m+1}) \le 0.$$

Then,

$$\Delta w_m \le\ -\frac{\rho_m q_{m+1} f(x_{m+1})}{x_{m+1}} + \frac{\Delta x_{m+1} c_{m+1} \Delta\rho_m}{x_{m+1}} - \frac{\rho_m c_m (\Delta x_m)^2}{x_m\ x_{m+1}}$$

In view of monotonicity of x_m and Δx_m , (O_{11}) we see that for $m \ge N$

$$\Delta w_m \le -\mathrm{L}\ \rho_m q_{m+1} + w_{m+1} \frac{\Delta\rho_m}{\rho_{m+1}} - w_{m+1}^2 \frac{\rho_m}{\rho_{m+1}^2},$$

where

$$-w_{m+1}^2 \frac{\rho_m}{\rho_{m+1}^2} + w_{m+1} \frac{\Delta\rho_m}{\rho_{m+1}} =$$

$$-\left(w_{m+1}^2 \frac{\rho_m}{\rho_{m+1}^2} -\ w_{m+1} \frac{\Delta\rho_m}{\rho_{m+1}}\right)$$

$$-\frac{\rho_m}{\rho_{m+1}^2}\left(w_{m+1}^2 - \frac{\Delta\rho_m \rho_{m+1}}{\rho_m}\ w_{m+1}\right)$$

$$= -\frac{\rho_m}{\rho_{m+1}^2}\left(w_{m+1}^2 - 2\ w_{m+1}\ \frac{\Delta\rho_m \rho_{m+1}}{2\rho_m} + \left(\frac{\Delta\rho_m \rho_{m+1}}{2\rho_m}\right)^2 - \left(\frac{\Delta\rho_m \rho_{m+1}}{2\ \rho_m}\right)^2\right)$$

$$-\frac{\rho_m}{\rho_{m+1}^2}\left(\ w_{m+1} - \frac{\Delta\rho_m \rho_{m+1}}{2\rho_m}\right)^2 + \frac{\rho_m}{\rho_{m+1}^2}\ \frac{(\Delta\rho_m)^2\ \rho_{m+1}^2}{4{\rho_m}^2}$$

$$-\frac{\rho_m}{\rho_{m+1}^2}\left(\ w_{m+1} - \frac{\Delta\rho_m \rho_{m+1}}{2\rho_m}\right)^2 + \frac{(\Delta\rho_m)^2}{4\rho_m}$$

Hence,

$$\Delta w_m \leq -\mathrm{L}\rho_m q_{m+1} - \frac{\rho_m}{\rho_{m+1}^2}\left(w_{m+1} - \frac{\Delta\rho_m \rho_{m+1}}{2\rho_m}\right)^2 + \frac{(\Delta\rho_m)^2}{4\rho_m}, \quad (m \geq 1)$$

$$\Delta w_m \leq -\mathrm{L}\rho_m q_{m+1} + \frac{(\Delta\rho_m)^2}{4\rho_m}, \quad (m \geq N).$$

When we add the aforementioned inequality from N to m, we obtain

$$w_{m+1} \leq w_N - \mathrm{L}\sum_{s=N}^{\infty} \rho_s \left[q_{s+1} + \left(\frac{\Delta\rho_s}{2\rho_s}\right)^2 \right]$$

Now by (C_{15}) it is easy to see that w_m is essentially negative which is a contradiction.

Remark 4.2.2 In Theorem 4.2.2, let $\theta_m = 0$, then it reduces to theorem 4 of (Thandapani, 1992).

Example 4.2.4 Consider the difference equation

$$\Delta(\Delta x_m) + \frac{1}{m+2}\Delta x_m + \frac{(4m^2+6m+1)}{m^{\frac{1}{3}}(2m+1)^{\frac{2}{3}}} x_{m+1} = 0,$$

Where

$c_m = 1, \theta_m = \frac{1}{m+2}, \; \varphi(x_m, \Delta x_m) = \Delta x_m, \; q_{m+1} = \frac{(4m^2+6m+1)}{m^{\frac{1}{3}}(2m+1)^{\frac{2}{3}}}, f(x) = x,$

Condition (C_1) is hold because

c_m=1

$$\theta_m = \frac{1}{m+2} > 0\,.$$

$$q_{m+1} = \frac{(4m^2+6m+1)}{m^{\frac{1}{3}}(2m+1)^{\frac{2}{3}}} > 0$$

Because the fraction is positive for all $m \in \mathbb{N}$.

(C_2) is hold because

$\varphi(x_m, \Delta x_m) = \Delta x_m \to \varphi(x_m, \Delta x_m)\Delta x_m = (\Delta x_m)^2 \leq (\Delta x_m)^2, L = 1,$

(C_3) is hold because

$x\,f(x) = x^2 > 0$ for each $x \neq 0$.

Condition (C_4) is hold because

$$c_m - L\theta_m = 1 - \frac{1}{m+2} > 0, \quad \forall\, m > 1,$$

Therefore, for a positive constant L, the condition is satisfied all m > 1where $L = 1$.

$\frac{f(x)}{x^\alpha} = \frac{x}{x} = 1 \geq L > 0$ for all $x \neq 0$

Therefore, for a positive constant L, the condition is satisfied**.**

We take $\rho_s = s$

Where $\Delta\rho_s = \rho_{s+1} - \rho_s = 1$

Therefore

$$\left(\frac{\Delta\rho_s}{2\rho_s}\right)^2 = \left(\frac{1}{2s}\right)^2 = \frac{1}{4s^2}$$

$$\sum_{s=N}^{\infty} s\left[\frac{(4s^2+6s+1)}{s^{\frac{1}{3}}(2s+1)^{\frac{2}{3}}} - \frac{1}{4s^2}\right] = \infty$$

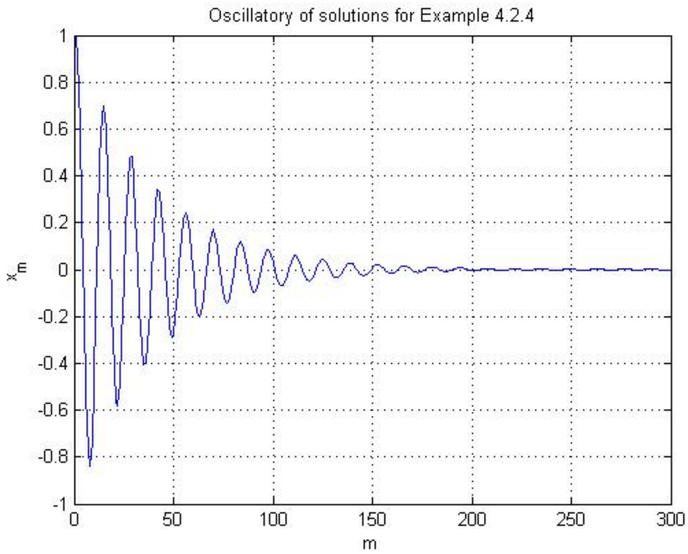

Fig 4.4: Oscillatory of solutions for example 4.2.4

Theorem 4.2.5 In addition to conditions $(C_1) - (C_3)$, (C_{10}) and (C_{11}) are hold. Assume that

$$(C_{16}) \sum_{m=N_0}^{\infty} q_{m+1} f(m) = \infty \quad , N_0 \in \mathbb{N}$$

All of equation *(E)* is solutions are oscillatory.

Proof. Consider the non-oscillator solution of equation *(E)* to be $\{x_m\}$.

Considering Lemma 4.2.1, it is permissible to suppose the existence of an integer $N_1 \geq N_0 \in \mathbb{N}$ such that $x_m > 0$ and $\Delta x_m > 0 \;\forall\; m \geq N_1$ without loss of generality.

Since,

$$0 < x_m\, \varphi(x_m, \Delta x_m).$$

So,

$$\varphi(x_m, \Delta x_m) > 0 \ \forall \ m \geq N_1.$$

Now,

$$\Delta(c_m x_m) + q_{m+1} f(y_{m+1}) \leq 0. \tag{4.13}$$

Since f is non-decreasing function, from Lemma 4.2.1, we have

$$f(x_{m+1}) \geq f\left(\frac{m\Delta x_m}{2}\right) \quad \text{for each } m \geq 2N_1. \tag{4.14}$$

Using (4.14) in (4.13) and then applying the condition (C_{10}) , we obtain

$$\Delta(c_m x_m) + \gamma f\left(\frac{1}{2}\right) q_{m+1} f(m) f(\Delta x_m) \leq 0$$

or

$$\frac{\Delta(c_m x_m)}{f(\Delta x_m)} + \gamma f\left(\frac{1}{2}\right) q_{m+1} f(m) \leq 0 \text{ for each } m \geq 2N_1. \tag{4.15}$$

Note that,

$$\Delta x_m \geq y \geq \Delta x_{m+1}$$

We have then

$$f(y) \leq f(\Delta x_m)$$

And there for

$$\frac{\Delta(c_m x_m)}{f(y)} \leq \frac{\Delta(c_m x_m)}{f(\Delta x_m)} \tag{4.16}$$

Using (4.16) in (4.15) After adding up the ensuing inequality between $2N_1$ and m we obtain

$$\gamma\, f\left(\frac{1}{2}\right) \sum_{s=2N_1}^{m} q_{s+1}\, f(s) \leq \sum_{s=2N_1}^{m} \int_{\Delta x_{s+1}}^{\Delta x_s} \frac{dy}{f(y)} \leq \int_{\Delta x_{n+1}}^{\Delta x_{2N_1}} \frac{dy}{f(y)} < \infty$$

Which contradicts (C_{16}) and this proves the theorem.

Remark 4.2.3. In Theorem 4.2.5 if $\theta_n = 0$, then it reduces to Theorem 3 of (Thandapani, 1992).

Example 4.2.5 Assuming we utilized the identical example 4.2.3

Where

$$x\, f(x) = x^2 > 0 \; for\ each\ x \neq 0$$

$$\sum_{s}^{\infty} \left(\sum_{s=n_0}^{m} \frac{1}{s+1} \right)^{1/3} \frac{1}{\sqrt{s+1}} = \infty,$$

f is non-decreasing because

$\acute{f}(x) = \frac{1}{3}\, x^{-2/3}$ for each $x > 0$.

Every solution of *(E)* is oscillatory, and all of the presumptions of Theorem 4.2.5 are met.

Theorem 4.2.6 In addition to conditions $(C_1) - (C_5)$ and assume $f(v)$ is non-decreasing function for $v \neq 0$ such that

(C_{17}) $\int_{\varepsilon}^{\infty} \frac{dv}{f(v)} < \infty$ and $\int_{-\infty}^{-\varepsilon} \frac{dv}{f(v)} > -\infty$

for all $\varepsilon > 0$ and

$$(C_{18})\ \sum_{m=N}^{\infty} m q_{m+1} = \infty, \qquad n_0 \in \mathbb{N}.$$

Then all solutions of equation *(E)* are oscillatory.

Proof. Assume that equation *(E)* has a non-oscillatory solution $\{x_m\}$. Take an integer $N_1 \geq N_0 \in \mathbb{N}$ such that $x_m > 0$ and $\Delta x_m > 0\ \forall\ m \geq N_1$. After demonstrating Theorem 4.2.1, if we multiply inequality by $\frac{m+1}{f(x_{m+1})}$ and then sum from $2N_1$ to $m-1$, we obtain:

$$\sum_{s=2N_1}^{m-1}(s+1)\frac{\Delta(c_s \Delta x_s)}{f(x_{s+1})} + \sum_{s=2N_1}^{m-1}(s+1)q_{s+1} \leq 0. \qquad (4.18)$$

Applying the summation by parts formula to the first term in (4.18), we obtain

$$\frac{m\Delta x_m}{f(x_m)} - \frac{2N_1\, \Delta x_{2N_1}}{f\left(x_{2N_1}\right)} - \sum_{s=2N_1}^{m-1} \frac{\Delta x_s}{f(x_{s+1})} + \sum_{s=2N_1}^{m-1} \frac{s\Delta f(x_s)}{f(x_s) f(x_{s+1})} + \sum_{s=2N_1}^{m-1}(s+1)q_{s+1} \leq 0,$$

or

$$\frac{m\Delta x_m}{f(x_m)} - \int_{y_{2N_1}}^{y_m} \frac{dy}{f(y)} + \sum_{s=2N_1}^{m-1}(s+1)q_{s+1} \leq C, \qquad (4.19)$$

where the constant C is positive.

Letting $n \to \infty$ in (4.19), we obtain contradiction.

Example 4.2.6 Consider the difference equation

$$\Delta\big((m+1)\Delta x_m\big) + \frac{1}{m+1}\Delta x_m + \frac{1}{\sqrt{m+1}} x^{3/4} = 0, m \geq 1,$$

where

$c_m = (m+1) > 0,$

$\theta_m = \frac{1}{m+1} > 0,$

$$q_{m+1} = \frac{1}{\sqrt{m+1}} > 0$$

$c_m - L\theta_m = (m+1) - \frac{1}{m+1} > 0$, $\forall\, m \geq 1$.

Therefore, for a positive constant L, the condition is satisfied all m ≥ 1where $L = 1$. and (C_5) is also hold because

$$\sum_{n=N_0}^{\infty} \frac{1}{(m+1)} \left[\prod_{s=N_0}^{m-1} \left(1 - \frac{1}{(m+1)^2}\right) \right] = \infty$$

$$\int_{\varepsilon}^{\infty} \frac{dv}{f(v)} = \int_{\varepsilon}^{\infty} \frac{dv}{v^{3/4}} = \left[\frac{v^{-1/3}}{-1/3} \right]_{\varepsilon}^{\infty} = \frac{3}{\varepsilon^{1/3}} < \infty$$

The integral is finite. Similarly, it can be show that $\int_{-\infty}^{\varepsilon} \frac{dv}{f(v)} > -\infty$.

$$\sum_{m=N}^{\infty} \frac{m}{\sqrt{m+1}} = \infty,$$

All the conditions of Theorem 4.2.5 are hold, and every solution to *(E)* exhibits oscillatory behavior.

Remark 4.2.4

In theorem 4.2.5, if $\theta_n = 0$, then it reduces to theorem 2 of (Thandapani, 1992).

Corollary 4.2.1

Replace the conditions (C_{17}) and (C_{18}) with condition(C_{14}), where $\alpha \,\epsilon\, [1, \infty)$. As a result, every solution to equation *(E)* exhibits oscillatory behavior.

Chapter Five

Conclusion and Future Work

5.1 Introduction

In this chapter, the results obtained in the study will be summarized. The conclusions drawn from the current study are presented in the second section of this chapter. Finally, suggestions for future work related to the topics presented in the thesis are identified and discussed.

5.2 Conclusions

In this section has explored and summarized the main findings regarding the oscillatory behavior of second-order nonlinear difference equations. By developing rigorous theorems and supporting them with carefully constructed examples, we have established a comprehensive theoretical framework for identifying conditions under which all nontrivial solutions of such equations are guaranteed to oscillate. The methods applied especially Riccati type transformations and summation based inequalities prove to be effective tools in analyzing the qualitative behavior of solutions. These results lay the foundation for extended studies involving more complex structures, such as higher-order, delayed, or dynamic difference equations on time scales. Hence, the chapter not only concludes the current investigation but also opens avenues for meaningful future research.

5.3 Future Research Directions

1) Expanding the Scope of the Study

Time-Delay and Fractional Difference Equations:

Future research could investigate oscillation behavior in delay difference equations and fractional difference equations, which are essential in modeling systems with memory effects or non-local dynamics.

Higher-Order and Time-Varying Systems:

Extending the analysis to higher-order difference equations and systems with time-dependent parameters would enhance the applicability of the theory to more complex dynamical systems.

2) Novel Methodological Approaches

Artificial Intelligence Techniques:

Machine learning and deep learning models could be employed to predict oscillatory patterns in nonlinear difference equations, offering a data-driven complement to analytical methods.

Advanced Numerical Algorithms:

Developing finite element methods or adaptive numerical schemes for solving highly nonlinear difference equations could improve computational efficiency and accuracy.

3) Interdisciplinary Applications

Behavioral Economics & Chemical Engineering:

Applying the derived oscillation criteria to economic models with nonlinear feedback or chemical reaction networks could provide new insights into stability and periodicity in these fields.

Neural Networks & Epidemiological Models:

Investigating oscillations in artificial neural networks and infectious disease models would contribute to understanding complex dynamical behaviors in biological and AI-driven systems.

4) Refining Existing Theories

Stochastic Perturbations:

Exploring oscillation conditions under random noise or stochastic disturbances would make the theory more robust for real-world applications where uncertainty is inherent.

Generalized Nonlinear Functions:

Extending the results to multi-valued or discontinuous nonlinear functions could further broaden the theoretical framework.

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Appendix

Matlab code for Example 4.2.1

```
clc;
clear;

M = 100;
x = zeros(1, M+2);


x(1) = 1;
x(2) = -1;

for m = 2:M
cm = m + 3;
cmp1 = m + 4;
thetam = 1 / (m + 3);
qm = (m + 3)*(4*m^2 + 10*m + 5);
f = @(x) x + x^3;

delta_x_m = x(m) - x(m-1);

x_guess = x(m);
for k = 1:10
fx = f(x_guess);
dfx = 1 + 3*x_guess^2;

f_val = cmp1*(x_guess - x(m)) - cm*delta_x_m + thetam*delta_x_m + qm*fx;
df_val = cmp1 + qm * dfx;

x_guess = x_guess - f_val / df_val;
end

x(m+1) = x_guess;
end

m_vals = 1:M+1;
plot(m_vals, x(2:M+2), 'b.-', 'LineWidth', 1.5)
xlabel('m')
ylabel('x_m')
title('Oscillatory of Solutions for Example 4.2.1 ')
grid on
```

Matlab code for Example 4.2.2

```
clc;
clear;


M = 100;
```

```
x = zeros(1, M+2);


x(1) = 1;
x(2) = -1;

for m = 2:M
cm = m + 2;
cmp1 = m + 3;
theatam = 1 / (m + 2);
qm = ((4*m^2 + 8*m + 5)^(1/3) /(m+2)^(5/3));
f = @(x)  x^3;

delta_x_m = x(m) - x(m-1);

x_guess = x(m);
for k = 1:10
fx = f(x_guess);
dfx = 1 + 3*x_guess^2;

f_val = cmp1*(x_guess - x(m)) - cm*delta_x_m + theatam*delta_x_m + qm*fx;
df_val = cmp1 + qm * dfx;

x_guess = x_guess - f_val / df_val;
end

x(m+1) = x_guess;
end

m_vals = 1:M+1;
plot(m_vals, x(2:M+2), 'b.-', 'LineWidth', 1.5)
xlabel('m')
ylabel('x_m')
title('Oscillatory of Solutions for Example 4.2.2 ')

grid on
```

Matlab code for Example 4.2.3

```
clc;
clear;


M = 100;
x = zeros(1, M+2);
```

```
x(1) = 1;
x(2) = -1;

for m = 2:M
cm = m + 1;
cmp1 = m + 2;
theatam = 1 / (m + 1);
qm = 1/(sqrt(m+1));
f = @(x)  x^(1/3);

delta_x_m = x(m) - x(m-1);

x_guess = x(m);
for k = 1:10
fx = f(x_guess);
dfx = 1 + 3*x_guess^2;

f_val = cmp1*(x_guess - x(m)) - cm*delta_x_m + theatam*delta_x_m + qm*fx;
df_val = cmp1 + qm * dfx;

x_guess = x_guess - f_val / df_val;
end

x(m+1) = x_guess;
end

m_vals = 1:M+1;
plot(m_vals, x(2:M+2), 'b.-', 'LineWidth', 1.5)
xlabel('m')
ylabel('x_m')
title('Oscillatory of Solutions for Example 4.2.3 ')
grid on
```

Matlab code for Example 4.2.4

```
clc;
clear;


M = 100;
x = zeros(1, M+2);


x(1) = 1;
x(2) = -1;
```

```
for m = 2:M
cm = 1;
cmp1 = 2;
theatam = 1 / (m + 2);
qm = (4*m^2+6*m+1)/(m^(1/2)*(2*m+1)^(2/3));
f = @(x)  x;

delta_x_m = x(m) - x(m-1);

x_guess = x(m);
for k = 1:10
fx = f(x_guess);
dfx = 1 + 3*x_guess^2;

f_val = cmp1*(x_guess - x(m)) - cm*delta_x_m + theatam*delta_x_m + qm*fx;
df_val = cmp1 + qm * dfx;

x_guess = x_guess - f_val / df_val;
end

x(m+1) = x_guess;
end

m_vals = 1:M+1;
plot(m_vals, x(2:M+2), 'b.-', 'LineWidth', 1.5)
xlabel('m')
ylabel('x_m')
title('Oscillatory of Solutions for Example 4.2.4 ')
grid on
```

الملـــخـــص

تعد معادلات الفروق النظير المنفصل للمعادلات التفاضلية العادية. فبدلاً من وصف معدلات التغير المستمرة، تصف هذه المعادلات التغيرات بين قيم الدوال عند نقاط منفصلة، ويسمح هذا السلوك التذبذبي بتوسيع مفاهيم حساب التفاضل والتكامل والتحليل الرياضي لتشمل بيانات منفصلة، وهو أمر بالغ الأهمية في مجالات مثل علوم الحاسوب ومعالجة الإشارات الرقمية، حيث تكون البيانات منفصلة بطبيعتها. تسلط هذه الدراسة الضوء على السلوك التذبذبي لحلول بعض معادلات الفروق غير الخطية من الرتبة الثانية ذات معامل التخميد، نظراً لأهميتها الكبيرة في الرياضيات التطبيقية، والتحليل الديناميكي. وباستخدام أساليب مثل تحويلات نوع ريكاتي والمتباينات المنفصلة، تم الحصول على عدد من شروط التذبذب لفئة معينة من معادلات الفروق، وبالتالي تم توسيع بعض النظريات السابقة. وستدعم الأمثلة التي توضح النتائج الفرضيات المطروحة في هذه الدراسة.