وزارة التعليم العالي والبحث العلمي

جامعـــــــة ســــــرت

إدارة الدراسات العليا والتدريب

كلية العلوم

قسم الرياضيات

البرنامج الماجستير

رسالة ماجستير بعنوان

(حل المعادلات التفاضلية ذات الرتبة الكسرية باستخدام مصفوفة العمليات لمويجيات بعض متعددات الحدود- دراسة تحليلية)

قدمت هذه الرسالة كأحد متطلبات الحصول على درجة الإجازة العالية (الماجستير)

إعداد الطالبـــــة: سعاد مفتاح عمر أبوخزرانة

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Solving Fractional-order Differential Equations by Some Wavelet –Polynomial operational Matrix

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Declaration

I Suad Miftah Omar Abu Khazranah confirm that the work contained in this thesis / dissertation, unless otherwise referenced is the researcher's own work, and has not been previously submitted to meet requirements of an award at this University or any other higher education or research institution, I furthermore, cede copyright of this thesis / dissertation in favour of University of Sirt.

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Abstract

Fractional differential equations generalize classical derivatives and integrals using fractional derivative operators. These equations have gained increasing popularity in recent years due to their applications in diverse fields, including physics, engineering, and financial science.

This dissertation investigates a numerical method for solving fractional-order differential equations subject to initial or boundary conditions using Legendre wavelet optimization. The core idea of this approach is to approximate both the known and unknown functions within the target equation using Legendre wavelets. Leveraging the orthogonality property of Legendre polynomials and the operational matrices for ordinary and fractional derivatives (according to the Caputo definition of the fractional derivative), the fractional differential equation is transformed into a system of algebraic equations that can be readily solved. To validate the accuracy and efficiency of this method, several illustrative examples of initial and boundary value problems are presented. The obtained numerical results are compared with the exact (analytical) solutions, confirming the effectiveness of this method for solving fractional-order differential equations.

Keywords: Fractional Calculus; Shifted Legendre Wavelets; Operational Matrix; Fractional Differential Equations; Caputo Fractional Derivative.

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I dedicate this dissertation to my father spirit (Allah bless him), my mother,

my husband,

my daughter, my sisters and my dear brother.

Acknowledgements

Praise be to Allah, who has granted me the strength and guidance to complete this work. I hope it will be beneficial and contribute to the field.

I would like to express my deepest gratitude to my supervisor professor **Dr. Haniyah Saed,** for her invaluable support and guidance throughout this journey. I also extend my heartfelt appreciation to my late professors, **Dr. Emtair** and **Dr. Abdelhamid-** may Allah have mercy on them for their lasting impact and dedication to their students.

Furthermore, I sincerely thank all the faculty members of the **Mathematics Department** and the **Faculty of Science** for their contributions to my academic growth.

I pray that this research serves as a valuable contribution to knowledge and that Allah grants success to all in their endeavors.

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Notations

$\mathbb{N} = \{1, 2, 3, \dots \}$.

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

$\mathbb{Z}$: The set of integer numbers.

$\mathbb{R} = (-\infty, \infty)$.

$\mathbb{R}^+ = (0, \infty)$.

$\mathbb{C}$: A complex number.

$[x]$: The greatest integer less than or equal.

L^2 $(\mathfrak{R})$: Vector space of measurable, square-integrable one-dimensional function $f(x)$.

List of symbols

$\Gamma(x)$	gamma function.
$\beta(m,n)$	beta function.
$\mathrm{erf(t)}$	error function.
$E_\alpha(t)$	Mittag-Leffler function.
${}^{RL}_{a}D^{\alpha}_{t}$	Fractional differentiation in the left- handed Riemann Liouville.
${}^{RL}_{t}D^{\alpha}_{b}$	Fractional differentiation in the right- handed Riemann Liouville.
${}^{RL}_{t}I^{\alpha}_{a+}$	Riemann Liouville Fractional integral.
${}^{C}_{a}D^{\alpha}_{t}$	Caputo fractional derivative.
${}^{G-L}_{a}D^{\alpha}_{t}$	Grunwald –Letnikov Fractional Derivatives.
$P_n(\mathrm{x})$	Legendre polynomial.
$\phi(t)$	Mother wavelet.
$\Phi_{a,b}(t)$	Wavelet transform function.
a	Wavelet scaling factor.
b	Center location of the wavelet function.
$P_n(\mathrm{x})$	Legendre polynomial.
$P_n^*(x)$	shifted Legendre wavelets.

Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non- integer) order. Many famous mathematicians contributed to this theory over the years; this field is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order.

The fractional calculus may be considered an old and yet novel topic. The beginning of the fractional is considered to be Leibniz's letter to l'Hospital in 1695.

It is an old topic since, starting from some speculations of L. Eule (1730) and G.W. Leibniz (1695,1697), it has been developed up to nowadays. A list of mathematicians, who have provided important contributions up to the middle of our century, includes J.L. Lagrange (1772), P.S. Laplace (1812), J.B.J. Fourier (1822), J. Liouville (1832), B. Riemann (1887), A.K. Grunwald (1867), H. Weyl (1917), E.R. Love (1938-1996), H. Kober (1940), A. Erdelyi (1939-1965).

The use of fractional calculus in viscoelasticity, wave motion, thermal conduction problems, signal and image processing, and blood flow phenomena, also in the tautochrone problem, and is sometimes called the equal time problem (synchronization).

The number of scientific and engineering problems involving fractional calculus is already very large and still growing.

Fractional order differential equations have important applications in fields like physics, engineering, and biology, but they are more complicated than ordinary differential equations and difficult to solve using traditional methods. Exploring techniques for transforming FDEs into systems of algebraic equations using wavelet operational matrices.

The **significance of this research** stems from the increasing need for accurate and efficient numerical solutions to fractional differential equations, which are essential mathematical models used to describe systems with non-local properties and memory effects. Due to the difficulty of obtaining analytical solutions for such equations, developing numerical methods based on operational matrices is a crucial step toward simplifying the solution process. Utilizing wavelets constructed from Legendre polynomials provides a robust numerical framework characterized by orthogonality and high accuracy, which contributes to improving results and reducing numerical error. Thus, this work offers a valuable scientific contribution in the field of numerical analysis of fractional equations and can be applied in various physical and engineering contexts.

This research aims to develop an efficient numerical method for solving fractional differential equations by constructing operational matrices for fractional derivatives based on Legendre polynomial wavelets. It also seeks to examine the accuracy and effectiveness of the proposed method by applying it to well-known test equations and comparing the results with analytical solutions (when available) or previously developed numerical methods. The goal is to minimize numerical error and enhance the computational stability of the proposed technique.

Wavelet theory is very significant in science, engineering, and technology. In recent years wavelets have achieved enormous attention in many fields of investigation, such as spectroscopy, signal analysis, feature detection in earth science, time-frequency analysis, image manipulation, etc. Many scholars have contributed to the development of wavelets. Especially, Daubechies, Belkin, Meyer, and Mallat are some of them. There has been a substantial increment in the number of studies on wavelets. There are a variety of wavelet functions such as Daubechies, Haar, Laguerre, Legendre, Shan-non, Lagrange, Hermitian, and Chebyshev wavelets. Among these wavelets, we chose Legendre and Bernoulli wavelets in this dissertation because of their orthonormality.

The Legendre wavelet function was chosen in this research for several advantages that make it a powerful tool for constructing operational matrices. One of the main advantages is its orthogonality, a property that helps improve numerical accuracy and reduces error when applied to fractional differential equations. By using orthogonal functions, interference between different functions is minimized, resulting in more accurate and less scattered results. In addition, Legendre functions have mathematical simplicity, making them easy to derive and use in complex calculations without the need for intricate operations. This simplicity makes them well-suited for numerical methods. Moreover, these functions offer high accuracy in approximating fractional derivatives, leading to improved numerical solution results and reducing the overall error. With these advantages, Legendre wavelets are considered an ideal choice for efficiently and effectively solving fractional differential equations.

This study is limited to fractional differential equations that involve linear functions only, with Legendre wavelets being used based on their orthogonality property to achieve accurate results. Nonlinear functions or nonlinear effects that may arise in certain physical or engineering applications have not been addressed in this research. Therefore, the results obtained are based on the fundamental assumption that the equations dealt with are linear.

Based on the above, and in order to clarify the foundation on which this work is built, the following section provides a review of the related literature, from these studies:

In this paper **(Belgacem et al., 2018)**, a numerical method based on Bernoulli polynomials is presented for numerical solutions of fractional differential equations. The Bernoulli operational matrix is derived and used with the Tau collection method to transform it into a system of linear algebraic equations. Therefore, the advantage presented by this paper is that it has a higher computational complexity and provides solutions with high accuracy. Building on this background, this paper headings a similar mode but with a new function to highlight the behavior of the solution under this adjustment. In the references **(Secer et al., 2019), (Chang et al., 2016) and (Isah & Chang, 2015)**, the Legendre wavelet operational matrix method for fractional matrices was presented which is a new method that combines fractional differentiation with wavelets, the Legendre function, and operational matrices. The studies propose a new technique based on Legendre waves to obtain better results using the properties of wavelets and orthogonal Legendre polynomials. Using Legendre wavelets for fractional matrices, approximating the solution and converting the problem to linear equations. The results demonstrated that this technique is effective and applicable. In addition, this method was developed to solve initial and boundary value problems for diffusion equations, and effective results were obtained and proven to be efficient. In contrast to the three previous studies, the present paper employs the Bernstein fractional differentiation operational matrix **(Saadatmandi, 2014)** was derived to the order alpha, and positive and accurate results were obtained. From here we notice that fractional matrices have become important, their studies are effective, and their results are accurate. It should be mentioned that, all these references use Caputo's fractional definition in deriving operational matrices.

The thesis is organized as follows:

- **Chapter One:**

This chapter, provides an introduction to fractional calculus, its history, and the development of its main concepts, with a focus on the fundamental definitions of fractional derivatives such as the Riemann–Liouville, Caputo, and Grünwald–Letnikov definitions. It also includes an explanation of some special functions related to fractional calculus, such as the Gamma, Beta, and Error functions, along with their properties. General properties of fractional derivatives are discussed and supported with illustrative examples. Additionally, the chapter introduces the concepts of the inner product and linear functions, as a foundation for the subsequent chapters.

- **Chapter Two:**

This chapter, aims to study Legendre polynomials to clarify and gain a deeper understanding of them. It includes the (differential equation, the generating function, its recurrence formula, the properties of Legendre polynomials, the orthogonal function and some examples). This will be used in research especially in chapter three.

- **Chapter Three**:

  This Chapter, provides an introduction to wavelets and their historical development, along with a definition of the Wavelet Transform and the most commonly used wavelet types. The chapter also discusses key properties of wavelets, such as orthogonality and normalization, due to their importance in numerical applications. Within this context, special attention is given to Legendre wavelets, particularly the Shifted Legendre Wavelets, as they are directly related to the subject of this study. The differences between the standard and shifted Legendre wavelets are clarified in terms of their mathematical structure and applications**.**

- **Chapter Four:**

  Chapter four presents the numerical solution of fractional differential equations using the Shifted Legendre Wavelet method. This chapter discusses the key components of the method, including the operational matrix of derivative and fractional derivative, as well as the normalized Shifted Legendre operational matrix of fractional-order derivatives. The approach used for solving linear fractional differential equations is detailed, and some example problems are solved using this technique. MATLAB was employed to implement the computational algorithms and generate the results, which were shown to be accurate and efficient.

- **Chapter Five:**

  A conclusion and future work are drawn in this chapter.

Chapter One

Fractional Calculus

Introductory Concepts

In this chapter, we present some special functions such as (Gamma, Beta, error function and Mittag –Leffler function) these functions play an important role in the theory of fractional derivatives. We also study the most commonly used definitions in fractional differentiation, which are Grünwald – Letnikov Fractional Derivatives, Riemann Liouville Fractional Derivatives and Caputo Fractional Derivative. Some examples are studied to illustrate how to use these functions.

1. **Introduction to Fractional Calculus** (Malinowska et al., 2015) and (Podlubny, 1998)

Fractional Calculus (FC) is a branch of mathematics that began three centuries ago. It is referred to as (FC) and there are other names for it, including generalized calculus and arbitrary calculus.

It can be considering an ancient and modern topic at the same time. It dates back to the seventeenth century through Leibniz and Euler and has witnessed gradual growth and progress in the present time, despite its origin as ancient as classical calculus itself.

It has several fields in engineering, physics and other sciences. Compared to the classical theory of differential equations, work on the theory of fractional equations is still in its beginnings.

2. **Historical Review** (Podlubny, 1998)**,**(Malinowska et al., 2015)

The concept of fractional derivatives appeared for the first time in a conversation between G.A.de L'Hospital and G.W. Leibniz in date 30 September 1695.

At that time, L'Hospital asked Leibniz:" what does $\frac{d^n}{dx^n} f(x)$ is mean if $n = \frac{1}{2}$ not an integer (rational number)? what could it be? Leibniz replied that it leads to a paradox from which useful results will someday.

Fractional Calculus is very old since its classical existence, but it has been developed in recent years by many authors and from them L. Euler (1730), J.L. Lagrange (1772), P.S. Laplace (1812), J.B. J. Fourier (1822*)*, J. Liouville (1832-1873), B. Riemann (1847), H. L. Greer (1859), A. K. Grünwald (1867-1872), H. Weyl (1919), A. Marchaud (1927), D. V. Widder (1941), W. Feller (1952) and others.

Recently, fractional differential equations (FDEs) have been of considerable interest.

In the received there are many applications such as: the use of fractional calculus in viscoelasticity, wave motion, thermal conduction problems, signal and image processing, and blood flow phenomena. One of the important of these applications is the Niels Heinrik Abel (1823-1826) application, which is tautochrone problem, and is sometimes called the equal time problem (synchronization).

All these developments have stimulated the study of FDEs, a topic that had for many years been a relatively mysterious field of mathematics. Most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. Recently, several numerical methods have been implemented to solve fractional differential equations.

Over the years, fractional calculus became a very attractive topic for mathematics and many different forms of fractional calculus operators were introduced, and continue to be to this day.

3. Some Special Functions:

In this section we provide some definitions and give certain features of some well-known functions such as the, Gamma, Beta, error function and Mittag-Leffler function.

- **The Gamma Function(1729) :**(Podlubny, 1998)**,** (Delkhosh, 2013)**,** (Askey & Roy, 2010)**,** (Dragomir et al., 1999)**,** (Bologna, 2005) and (Farrell & Ross, 2013)

The gamma function is an extension of the multiplication function for real and complex numbers. It was first introduced by the Swiss mathematician Euler in 1707.

It is symbolized by the Greek Γ symbol, and it plays a fundamental role in fractional calculation.

- **Definition 1.1:** (Farrell & Ross, 2013)**,** (Dragomir et al., 1999)**,** (Bologna, 2005)

The Euler definition of gamma function is widely used in mathematician and helps in solving many problems. It is most common and is known as the following:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \, , x > 0, x \in \mathbb{R}^+ \tag{1.1}$$

where,

$\Gamma(x)$ is gamma function, $\mathbb{R}^+ = (0, \infty)$ is the set of positive real numbers**.**

Definition (Euler) 1. 2: (Dragomir et al., 1999)**,** (Bologna, 2005)

Using the infinite product gives Euler's definition of the gamma function as follows:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)(x+2) \ldots . (x+n)}$$

For example, $x = 1$

$$\Gamma(1) = \lim_{n\to\infty} \frac{n!\ n}{(1)(2)(3) \ldots (1+n)}$$

$$\Gamma(1) = \lim_{n\to\infty} \frac{n!\ n}{(1+n)!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)$$

$$\Gamma(1) = 1.$$

Definition (Weierstrass) 1.3: (Podlubny, 1998)**,** (Delkhosh, 2013)**,** (Askey & Roy, 2010)**,** (Dragomir et al., 1999)

It is another way to present the gamma function

$$\Gamma(x)\Gamma(-x) = \frac{-\pi}{x sin(\pi x)}$$

There are many other definitions of the gamma function.

Some Properties of Gamma Function:

1) $\Gamma(1) = 1$
2) $\Gamma(2) = 1\Gamma(1) = 1!$
3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
4) $\Gamma(n) = (n-1)!, n = 1,2,3, \ldots$
5) $\Gamma(n+1) = n\Gamma(n)$
6) $\Gamma\left(n+\frac{1}{2}\right) = \frac{2n!}{2^{2n} n!}\sqrt{\pi}$

Example 1. 1: -

1) $\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$

2) $\Gamma\left(-\frac{3}{2}\right)=\frac{\Gamma\left(\frac{-1}{2}\right)}{\left(\frac{-3}{2}\right)}=\frac{1}{\frac{-3}{2}}\times\frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-1}{2}}=\frac{4}{3}\sqrt{\pi}$

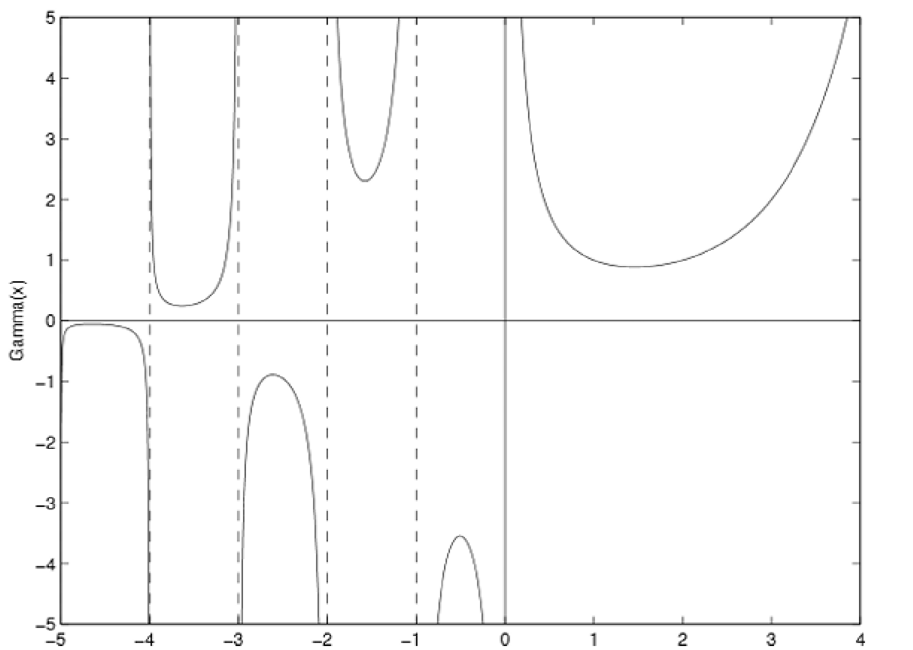


Figure 1.1: Gamma Function for Real Numbers

- **The Beta Function:** (Podlubny, 1998)**,** (Delkhosh, 2013)**,** (Dragomir et al., 1999) and (Bologna, 2005)

  This name was used in 1990 by Whittaker, Legendre and Watson, and it is symbolized by the symbol $\beta(z,w)$,which is a Greek letter.

  It is an important mathematical tool and is used in many applications, as z,w is two real numbers greater than zero.

  Definition 1. 4: (Podlubny, 1998)**,** (Delkhosh, 2013)**,** (Dragomir et al., 1999) and (Bologna, 2005)

  Beta function is defined by the integral

$$\beta(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt\,, Re(\,z)>0, Re\,(w)>0\,, w\in\mathbb{R}^+ \qquad (1.2)$$

Where β is beta function, and $\mathbb{R}^+$is the set of positive real numbers.

Relation Between the Gamma and Beta Function is;

$$\beta(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)} \qquad (1.3)$$

Remark 1.1*:* The Beta function is symmetric as:

$$\beta(z, w) = \beta(w, z)$$

Example 1. 2:

$$\beta(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{1}{12}$$

$$\beta(3,2) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{1}{12}$$

- **The Error Function ($\boldsymbol{erf}$)** (Kimeu, 2009) and (Chevillard, 2012)

Definition 1. 5: (Kimeu, 2009) and (Chevillard, 2012)

The error function is defined as

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \ , t > 0 \qquad (1.4)$$

A function which is closely related to the error function is

The Complementary Error Function ($\boldsymbol{erfc}$) is defined as:

$$erfc(t) = 1 - erf(t)$$

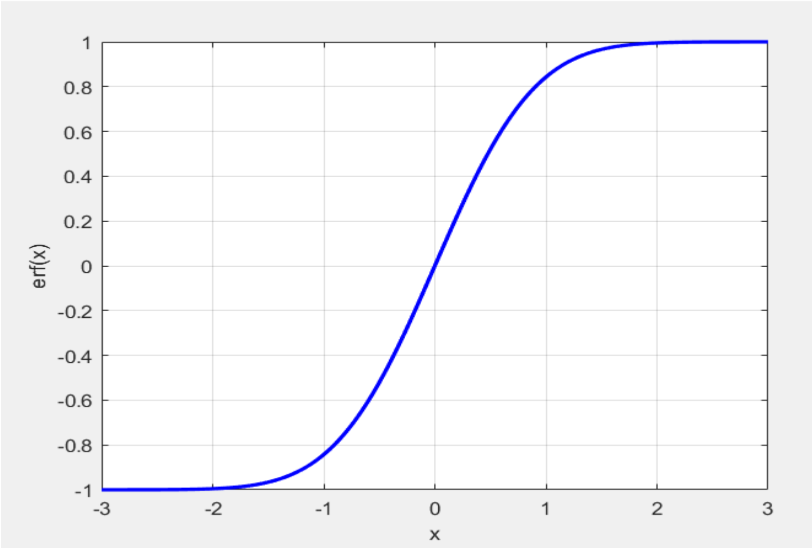


Figure 1. 2: The Error Function (erf)

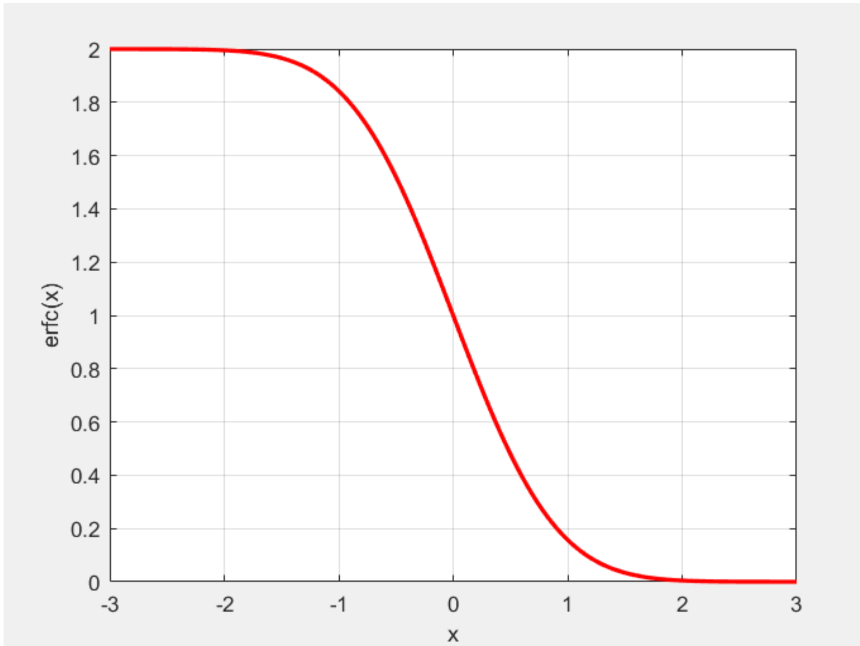


Figure 1.3: The Complementary Error Function ($erfc$)

Example 1.3:

1) $erf(0) = 0$
2) $erf(\infty) = 1$
3) $erfc(t) = 1 - erf(t) \Longrightarrow erfc(t) + erf(t) = 1$

Remark 1.2: (Kimeu, 2009) and (Chevillard, 2012)

The error function is an odd function

$$erf(-t) = -erf(t)$$

and also symmetric about the origin as in the figure above.

- **The Mittag – Leffler Function:** (Podlubny, 1998) and (Atanackovic, 2013)

The Mittag-Leffler function introduced by the mathematician Gosta Mittag-Leffler in 1903 generalization of the exponential function, and it is one of the essential functions related to fractional differential equations. The interest in the Mittag-Leffler function increased due to its importance and applications in non-integer order calculus, the so-called fractional calculus, and fractional order differential equations.

Definition 1.6: (Wiman, 1905)

(Mittag-Leffler function)

1. In 1905, Wiman gives of one parameter is given by

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} \;\; ; \;\; \alpha > 0 \in \mathbb{R}, t \in \mathbb{C} \tag{1.5}$$

2. In 1971, Prabhakar introduced new of two parameters is given by (Prabhakar, 1971)

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)} \;\; ; \;\; \alpha, \beta > 0 \in \mathbb{R}, t \in \mathbb{C} \tag{1.6}$$

Examples 1.4: (Podlubny, 1998)

Some properties of Mittag- Leffler function from equation (1.6) showed as:

$$1)\; E_{1,1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = \sum\nolimits_{k=0}^{\infty} \frac{t^k}{k!} = e^t$$

where,

$$e^t = \sum\nolimits_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

$$2)\; E_{1,2}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} = \frac{t}{t} \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} = \frac{1}{t} \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)}$$

Let $k + 1 = n$

$$E_{1,2}(t) = \frac{1}{t} \sum_{n=1}^{\infty} \frac{t^n}{n!} = \frac{1}{t}\left\{t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right\} = \frac{1}{t}\left\{-1 + 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right\}$$

$$E_{1,2}(t) = \frac{1}{t}\{-1 + e^t\} = \frac{e^t - 1}{t}$$

$$3)\; E_{2,1}(t^2) = \sum_{k=0}^{\infty} \frac{t^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2\mathrm{k}!} = \cosh(t)$$

where,

$$\cosh(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots$$

$$4)\ E_{2,2}(t^2) = \sum_{k=0}^{\infty} \frac{t^{2k}}{\Gamma(2k+2)} = \frac{t}{t}\sum_{k=0}^{\infty} \frac{t^{2k}}{(2\mathrm{k}+1)!} = \frac{1}{t}\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2\mathrm{k}+1)!} = \frac{\sinh(t)}{t}$$

where,

$$\sinh(t) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$$

Lemma 1. 1: (Podlubny, 1998)

When $\beta = 1$ we obtain the one parameter Mittag –Leffler function instead of two parameter Mittag –Leffler function.

$$E_{\alpha,1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)} = E_{\alpha}(t)$$

4. Fractional Derivative

There are many important definitions in the fractional derivative, the most important of which is:

4.1. Grünwald – Letnikov Fractional Derivatives: (Podlubny, 1998), (Atanackovic, 2013) , and (Ray, S. S., & Gupta, A. K. ,2018)

Definition1.7: (Podlubny, 1998), (Atanackovic, 2013) and(Ray, S. S., & Gupta, A. K. ,2018)

The Grünwald– Letnikov [A. K. Grünwald (1867)] fractional derivative of order $\alpha > 0$ is defined as:

$$^{G-L}_{\ \ a}D_t^{\alpha} f(t) = \lim_{\substack{h\to 0\\ nh=t-\alpha}} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t-rh),\ \ t > a\,, t \geq 0 \qquad (1.7)$$

Where $n = \left[\frac{t-\alpha}{h}\right]$ and $\binom{\alpha}{\mathrm{r}}$ it represents the binomial coefficients of a non-integer number, we get:

$$\binom{\alpha}{\mathrm{r}} = \frac{\Gamma(\alpha+1)}{\Gamma(r+1)\Gamma(\alpha-r+1)} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-r+1)}{r!}, r = 1,2,\ldots$$

The Grünwald– Letnikov definition is simply a generalization of the ordinary discretization formula for integer order derivatives, we have

$$\frac{df(t)}{dt} = \lim_{h\to 0} \frac{f(t)-f(t-h)}{h}$$

4.2.The Riemann Liouville Fractional Integral and Derivatives (Podlubny, 1998), (Atanackovic, 2013) and (Ghosh et al., 2015)

In this part, we will talk the Riemann Liouville Fractional integral and fractional derivatives.

Definition 1.8: (Podlubny, 1998), (Atanackovic, 2013) and (Ghosh et al., 2015)

The Riemann-Liouville definition as:

$$ {}_{a}^{RL}I_{t}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau\ ,t\in[a,b]\,,\alpha>0,\alpha\in\mathbb{R}^{+} \tag{1.8}$$

In the special case when $f(t)=t^{\beta}$, we have

$$ {}_{a}^{RL}I_{t}^{\alpha}(t)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\alpha+\beta}\,,(\alpha>0,\beta>0) \tag{1.9}$$

Where $\Gamma(\alpha)$ is the well-known gamma function, $\mathbb{R}^{+}$is the set of positive real numbers.

- **Some Properties Riemann - Liouville Fractional Integral as Follows:**
  1. $({}^{RL}I^{\alpha}\,{}^{RL}I^{\beta})f(t)={}^{RL}I^{\alpha+\beta}f(t),(\alpha>0,\beta>0)$
  2. ${}^{RL}I^{\alpha}\,{}^{RL}D^{\alpha}f(t)=f(t)$

For example:

Use the above equation (1.9) with we α =1 and $\beta=0.5$ find the Riemann – Liouville's fractional integral of the following functions:

$$I^{1}(t^{0.5})=\frac{\Gamma(1.5)}{\Gamma(2.5)}t^{1.5}=\frac{\frac{\sqrt{\pi}}{2}}{\frac{3\sqrt{\pi}}{4}}t^{1.5}=\frac{2}{3}t^{1.5}$$

Remark 1.3: (Podlubny, 1998) and (Atanackovic, 2013)

When $\alpha=1,$ the fractional Riemann - Liouville integral returns to ordinary integral, meaning it returns to the classical integral.

For example,

When $\alpha=1,\beta=0$,we find that (1.9)

$$I^{1}(1)=\frac{\Gamma(1+0)}{\Gamma(0+1+1)}t^{0+1}=\frac{\Gamma(1)}{\Gamma(2)}\ t=t$$

this is perfect in classical integration

$$\int 1\ dt=t+c$$

Riemann Liouville Fractional Derivatives (RLFD):

The Riemann–Liouville integral is introduced by Bernhard Riemann and Joseph Liouville in1832. Perhaps, it is the first definition of fractional derivative that allows for a non-zero value for fractional differentiation of the constant.

Definition 1. 9: (Podlubny, 1998), (Atanackovic, 2013) and (Ghosh et al., 2015)

Suppose that $\alpha, a, t, b \in \mathbb{R}^+$ where$(n-1 \leq \alpha < n)$ then

1) **Fractional Differentiation in the Left- Handed Riemann Liouville Concept:**

$$ {}_{a}^{RL}D^{\alpha} f(t) = D^{n} I^{n-\alpha} f(t) $$

$$ {}_{a}^{RL}D_{t}^{\alpha}\, f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{a}^{t} \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}\,, \qquad \forall\, t > 0 $$

2) **Fractional Differentiation in the Right- Handed Riemann Liouville Concept:**

$$ {}_{t}^{RL}D_{b}^{\alpha}\, f(t) = (-1)^{n} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t}^{b} \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}\,, \forall t > 0 $$

Definition 1. 10: (Podlubny, 1998) and (Atanackovic, 2013)

The Riemann-Liouville definition in general is:

$$ {}_{a}^{RL}D_{t}^{\alpha}\, f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{a}^{t} (t-\tau)^{n-\alpha-1} f(\tau) d\tau\,, (n-1 \leq \alpha < n),\ \forall t > 0 \qquad (1.10) $$

Or

$$ {}_{a}^{RL}D_{t}^{\alpha}\, f(t) = \frac{d^n}{dt^n}\, {}_{a}^{RL}D_{t}^{-(n-\alpha)}\, f(t), (n-1 \leq \alpha < n), \forall t > 0 \qquad (1.11) $$

Theorem 1.1: (Podlubny, 1998) and (Atanackovic, 2013)

Riemann-Liouville fractional derivative${}_{a}^{RL}D_{t}^{\alpha}\, f(t)$ of the power function $f(t) = (t-a)^{v}$**,**where v is a real number is

$$ D^{\alpha}(t-a)^{v} = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} (t-a)^{v-\alpha}; v \in \mathbb{R} \qquad (1.12) $$

Proof

Let $f(t) = (t-a)^{v}$, we find that, From the equation (1.10)

$$ {}_{a}^{RL}D_{t}^{\alpha}(t-a)^{v} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{a}^{t} (t-\tau)^{n-\alpha-1} (\tau-a)^{v} d\,\tau $$

let $\tau = a + y(t-a), when\, \tau = a \Rightarrow y_1 = 0, when\, \tau = a \Rightarrow\ y_2 = 1\,, d\tau = (t-a)dy$

$$ {}_{a}^{RL}D_{t}^{\alpha}(t-a)^{v} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{0}^{1} (t-(a+y(t-a)))^{n-\alpha-1} (a+y(t-a)-a)^{v} (t-a)dy $$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^1 (1-y)^{n-\alpha-1} (t-a)^{n-\alpha-1}\, y^v (t-a)^v (t-a) dy$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} (t-a)^{v+n-\alpha} \int_0^1 (1-y)^{n-\alpha-1}\, y^v dy$$

$${}_{a}^{RL}D_t^{\alpha}(t-a)^v = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} (t-a)^{v-\alpha}$$

Important comment:

From the equation (1.2), we get

let $z-1 = n-\alpha-1 \Rightarrow z = n-\alpha\, , w-1 = v \Rightarrow w = v+1$

$$\int_0^1 (1-y)^{n-\alpha-1}\, y^v dy = \frac{\Gamma(n-\alpha)\Gamma(v+1)}{\Gamma(v+n-\alpha+1)}$$

where

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\frac{\Gamma(n-\alpha)\Gamma(v+1)}{\Gamma(v+n-\alpha+1)} (t-a)^{v+n-\alpha} \right]$$

$$= \frac{\Gamma(v+1)}{\Gamma(v+n-\alpha+1)} \frac{d^n}{dt^n} (t-a)^{v+n-\alpha}$$

where,

$$\frac{d^n}{dt^n} (t-a)^{v+n-\alpha} = (v+n-\alpha)(v+n-\alpha-1) \ldots (v-\alpha+1)(t-a)^{v-\alpha}$$

$$= \frac{\Gamma(v+1)(v+n-\alpha)(v+n-\alpha-1) \ldots (v-\alpha+1)(t-a)^{v-\alpha}}{(v+n-\alpha)(v+n-\alpha-1) \ldots (v-\alpha+1)\Gamma(v-\alpha+1)}$$

Thus, one has

$${}_{a}^{RL}D_t^{\alpha}(t-a)^v = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} (t-a)^{v-\alpha}$$

Example 1. 5:

Use the above theorem (1.1) with $\alpha = 0$ and $\alpha = 0.5$ to find the Riemann-Liouville fractional derivatives of the following function:

(1) $f(t) = t^3$, with $\alpha = 0$ & $v = 3$

(2) $f(t) = t^3$, with $\alpha = 0.5$ & $v = 3$

Solution:

(1) Since $\alpha = 0$, so

$$ {}^{RL}_{a}D_t^{0}(t-0)^3 \ = \frac{\Gamma(3+1)}{\Gamma(3-0+1)}(t-0)^{3-0} = \frac{\Gamma(4)}{\Gamma(4)}t^3 = t^3 $$

(2) Since $\alpha = \frac{1}{2}$, so

$$ {}^{RL}_{a}D_t^{1/2}(t-0)^3 \ = \frac{\Gamma(3+1)}{\Gamma(3-1\backslash 2+1)}(t-0)^{3-\frac{1}{2}} = \frac{\Gamma(4)}{\Gamma(7/2)}t^{5/2} = \frac{6}{\frac{15}{8}\sqrt{\pi}}t^{5/2} $$

- **Some Properties of the Riemann-Liouville Fractional Derivative** (Podlubny, 1998)**,** (Atanackovic, 2013)

Property 1:

From equation (1.11), we substitute its values $\alpha = n-1$ then we obtain an order $(n-1)$

$$ {}^{RL}_{a}D_t^{n-1}f(t) = \frac{d^n}{dt^n}({}^{RL}_{a}D_t^{-(n-(n-1))}f(t) $$

$$ = \frac{d^n}{dt^n}\,{}^{RL}_{a}D_t^{-1}f(t) = f^{(n-1)}(t) $$

Property 2:

Using (1.11), For $\alpha = n \ \geq 1 \ and \ t > a$

$$ {}^{RL}_{a}D_t^{n}\, f(t) = \frac{d^n}{dt^n}({}^{RL}_{a}D_t^{0}f(t)) $$

$$ = \frac{d^n}{dt^n}f(t) = f^n(\mathrm{t}) $$

Property 3:

One of the most important properties of Riemann-Liouville fractional derivative, where $\alpha > 0 \ and \ t > a$ is that,

$$ {}^{RL}_{a}D_t^{\alpha}\,({}^{RL}_{a}D_t^{-\alpha}\, f(t)) = \ f(t) $$

Proof

For $\alpha = n \geq 1$, we have:

$$ {}^{RL}_{a}D_t^{\alpha}\,({}^{RL}_{a}D_t^{-\alpha}\, f(t)) = \frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-1}\, f(\tau)d\tau $$

$$ {}^{RL}_{a}D_t^{\alpha}\,({}^{RL}_{a}D_t^{-\alpha}\, f(t)) = \frac{d}{dt}\int_a^t f(\tau)\, d\tau = f(t) $$

Remark1.4:

$$ {}_{a}^{RL}D_{t}^{\alpha}\left[{}_{a}^{RL}D_{t}^{-\alpha}\ f(t)\right]\ = f(t). $$

In general, when $\alpha \geq \beta \geq 0\ and\ \beta \geq \alpha \geq 0$, then we have:

$$ {}_{a}^{RL}D_{t}^{\alpha}\left[{}_{a}^{RL}D_{t}^{-\beta}\ f(t))\right]\ = {}_{a}^{RL}D_{t}^{\alpha-\beta}\ f(t) $$

Property 4: linearity:

$$ {}_{a}^{RL}D_{t}^{\alpha}\big(\lambda f(t)+\mu g(t)\big) = \lambda\, {}_{a}^{RL}D_{t}^{\alpha} f(t) + \mu\, {}_{a}^{RL}D_{t}^{\alpha} g(t) $$

Where $n-1<\alpha<n, \alpha \in \mathbb{R}, \lambda$ and μ are constants, $f(t)$ and $g(t)$ are functions.

Property 5: Non-commutation:

$$ {}^{RL}D^{\alpha}D^{n}f(t) = {}^{RL}D^{\alpha+n}f(t) \neq D^{n\,RL}D^{\alpha}f(t) $$

Where $n-1<\alpha<n, \alpha \in \mathbb{R}\,, N \in \mathbb{N}$ and $f(t)$ is function.

Property 6: Interpolation:

1. $\lim\limits_{\alpha\to n} {}^{RL}D^{\alpha}f(t) = f^{n}(t)$

2. $\lim\limits_{\alpha\to n-1} {}^{RL}D^{\alpha}f(t) = f^{n-1}(t)$

Where $n-1<\alpha<n, \alpha \in \mathbb{R}\,, n \in \mathbb{N}$ and $f(t)$ is function.

Corollary 1.1: :(Podlubny, 1998) and (Atanackovic, 2013)

The Riemann –Liouville derivative of constant is not zero, that is:

$$ {}_{a}^{RL}D_{t}^{\alpha}\ C = \frac{C\ t^{-\alpha}}{\Gamma(1-\alpha)} \neq 0, c =\ constant $$

Proof

From equation (1.10), for $(a = 0)$ can be written as

$$ {}_{a}^{RL}D_{t}^{\alpha}\ C = \frac{C}{\Gamma(n-\alpha)}\ \frac{d^{n}}{dt^{n}}\int_{0}^{t}(t-\tau)^{n-\alpha-1}d\tau $$

$$ {}_{a}^{RL}D_{t}^{\alpha}\ C = \frac{d^{n}}{dt^{n}}\frac{C\ t^{n-\alpha}}{(n-\alpha)\Gamma(n-\alpha)} $$

$$ {}_{a}^{RL}D_{t}^{\alpha}\ C = \frac{C}{\Gamma(n-\alpha+1)}\ \frac{\Gamma(n-\alpha+1)}{\Gamma(1-\alpha)}\ t^{-\alpha} = \frac{C}{\Gamma(1-\alpha)}\ t^{-\alpha} \neq 0 $$

For example,

$$D^{\frac{1}{2}}(3) = \frac{3}{\Gamma(\frac{1}{2})} t^{-1/2} = \frac{3}{\sqrt{\pi t}}$$

4.3.Caputo Fractional Derivative (CFDs) (Podlubny, 1998)**,** (Ray & Gupta, 2018)and (Atanackovic, 2013)

The Riemann- Liouville (RLFD) definition played an important role in the development of the theory of fractional calculus. But the demands technology requires a certain revision of the well-established pure mathematical approach, especially in the theory of viscoelasticity, where fractional derivatives are used for better description of material properties. Applied problem require definitions of fractional derivative as the utilization of physically interpretable initial conditions, and the Caputo definition is convenient for that. The Caputo definition provides a compromise between the weak conditions on $f(x)$ and the undesirable of the Riemann- Liouville.

Definition 1. 11: (Ray & Gupta, 2018)

In 1967, the Italian Mathematician M. Caputo introduced the Caputo fractional derivative of a function $f(x)$, as

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{a}^{t} \frac{f^{n}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & , n-1 < \alpha < n \\ \dfrac{d^{n}}{dt^{n}} f(t) & , \quad \alpha = n \end{cases} \tag{1.13}$$

Where the parameter α is the order of the derivative and it can be real or complex in caputo and n is an integer.

Theorem 1.2: (Podlubny, 1998)

Caputo fractional derivative ${}_{a}^{C}D_{t}^{\alpha}\, f(t)$ of the power function $f(t) = t^{v}$;

$$ {}_{a}^{C}D_{t}^{\alpha} t^{v} = \begin{cases} \dfrac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} \; t^{v-\alpha} & , n-1 < \alpha < n, v > n-1, v \in \mathbb{R} \\ \quad 0 \quad , & n-1 < \alpha < n, v \leq n-1, \; v \in \mathbb{N} \end{cases} \tag{1.14}$$

Proof:

The first case is:

$$D^{\alpha}t^{v} = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\frac{f^{n}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau$$

When $a = 0$, we have

$$= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{(\tau^{v})^{n}}{(t-\tau)^{\alpha+1-n}}d\tau$$

Where,

$$(t^{v})^{n} = (vt^{v-1})^{n-1}$$
$$= (v(v-1)t^{v-2})^{n-2}$$

$$= v(v-1)(v-2)\ldots(v-n+1)t^{v-n}$$

$$= \frac{\Gamma(v+1)}{\Gamma(v-n+1)}t^{v-n}$$

$$= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{\Gamma(v+1)}{\Gamma(v-n+1)}(\tau^{v-n})(t-\tau)^{n-\alpha-1}d\tau$$

Now, if use the substitution $\tau = \lambda t$, $d\tau = t\, d\lambda$, where $0 \leq \lambda \leq 1$,then we have

$$D^{\alpha}t^{v} = \frac{\Gamma(v+1)}{\Gamma(n-\alpha)\Gamma(v-n+1)}\int_{0}^{1}(\lambda t)^{v-n}((1-\lambda)t)^{n-\alpha-1}\ t d\lambda$$

$$= \frac{\Gamma(v+1)}{\Gamma(n-\alpha)\Gamma(v-n+1)}t^{v-\alpha}\int_{0}^{1}(\lambda)^{v-n}(1-\lambda)^{n-\alpha-1}d\lambda$$

$$= \frac{\Gamma(v+1)}{\Gamma(n-\alpha)\Gamma(v-n+1)}t^{v-\alpha}\beta(v-n+1, n-\alpha)$$

By using the relation (1.3)

$$D^{\alpha}t^{v} = \frac{\Gamma(v+1)}{\Gamma(n-\alpha)\Gamma(v-n+1)}t^{v-\alpha}\frac{\Gamma(v-n+1)\Gamma(n-\alpha)}{\Gamma(v-\alpha+1)}$$

$$= \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)}\ t^{v-\alpha}$$

As for the second case, when $(t^{v})^{n} = 0, n-1 < \alpha < n, v \leq n-1,\ \ v \in \mathbb{N}$, this is done through the constant function, which has the derivative of the fractional caputo equal 0.

Example 1.6: Use the theorem (1.2) with $\alpha = \frac{1}{2}\ and\ \ \frac{1}{3}\ and\ v = 1\ and\ \ 2$ to find the CFDs of the following functions:

1. $f(t) = t, with\ \alpha = \frac{1}{2}, v = 1$

2. $f(t) = t^{2},\ with\ \alpha = \frac{1}{3}, v = 2$

Solution:

1. Since $\alpha = \frac{1}{2},\ then\ v = 1$, so

$$D^{\alpha}t = \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)}\ t^{v-\alpha}$$

$$D^{\frac{1}{2}}\mathbf{t} = \frac{\Gamma(1+1)}{\Gamma\left(1+1-\frac{1}{2}\right)}\ t^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}t^{\frac{1}{2}} = 2\frac{\sqrt{t}}{\sqrt{\pi}}$$

2. Since $\alpha = \frac{1}{3},\ then\ v = 2$, so

$$D^{\alpha}f(t) = \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)}\ t^{v-\alpha}$$

$$D^{\frac{1}{3}}t^2 = \frac{\Gamma(2+1)}{\Gamma\left(2+1-\frac{1}{3}\right)}\ t^{2-\frac{1}{3}} = \frac{\Gamma(3)}{\Gamma\left(\frac{8}{3}\right)}\ t^{\frac{5}{3}} \approx 1.33t^{\frac{5}{3}}$$

Example 1. 7: If $a = 0$, $\alpha = \frac{1}{2}, n = 1$, $f(t) = t$ then find the Caputo fractional derivative of order $\frac{1}{2}$ of function $f(t) = t$?

Solution: Form the equation (1.13), we find that

$$D^{\frac{1}{2}}t = \frac{1}{\Gamma(\frac{1}{2})}\frac{d}{dt}\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}.\tau\, d\tau$$

Suppose that

$u = (t-\tau)$ $, du = -d\tau, -du = d\tau$ and $\tau = (t-u)$ when $\tau = 0 \Rightarrow u = t$ and $\tau = t \Rightarrow u = 0$

$$D^{\frac{1}{2}}t = \frac{-1}{\sqrt{\pi}}\frac{d}{dt}\int_t^0 \frac{1}{u^{\frac{1}{2}}}.(t-u)\, du$$

$$= \frac{-1}{\sqrt{\pi}}\frac{d}{dt}\int_t^0 u^{-\frac{1}{2}}.(t-u)\, du$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dt}[-\int_t^0 u^{-\frac{1}{2}}t du + \int_t^0 u^{-\frac{1}{2}}u\, du\,]$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dt}[-2tu^{\frac{1}{2}}|_t^0 + \frac{2}{3}u^{\frac{3}{2}}|_t^0$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dt}\frac{4}{3}t^{\frac{3}{2}} = \frac{2\sqrt{t}}{\sqrt{\pi}}$$

Or

Let $u = (t-\tau)^{1/2} \Rightarrow u^2 = t - \tau \Rightarrow 2udu = -d\tau$

$$D^{\frac{1}{2}}t = \frac{-2u}{\sqrt{\pi}}\int_{\sqrt{t}}^0 \frac{1}{u}du$$

$$= \frac{2u}{\sqrt{\pi}}\int_0^{\sqrt{t}} \frac{1}{u}du$$

$$D^{\frac{1}{2}}t = \frac{2\sqrt{t}}{\sqrt{\pi}}$$

- **Some Properties of the Caputo Fractional Derivative** (Podlubny, 1998) and (Atanackovic, 2013)

Property 1: linearity:

$$ {}_{a}^{C}D_t^{\alpha}\big(\lambda f(t) + \mu g(t)\big) = \lambda\, {}_{a}^{C}D_t^{\alpha} f(t) + \mu\, {}_{a}^{C}D_t^{\alpha} g(t) $$

Where $n-1 < \alpha < n, \alpha \in \mathbb{R}, \lambda$ and μ are constants, $f(t)$ and $g(t)$ are functions.

Property 2: Non-commutation:

$$ {}^{C}D^{\alpha} D^{n} f(t) = {}^{C}D^{\alpha+n} f(t) \neq D^{n}\, {}^{C}D^{\alpha} f(t) $$

Where $n-1 < \alpha < n, \alpha \in \mathbb{R}\,, n \in \mathbb{N}$ and $f(t)$ is function.

Property 3: Interpolation:

1. $\lim\limits_{\alpha \to n} {}^{C}D^{\alpha} f(t) = f^{n}(t)$

2. $\lim\limits_{\alpha \to n-1} {}^{C}D^{\alpha} f(t) = f^{n-1}(t) - f^{n-1}(0)$

Where $n-1 < \alpha < n, \alpha \in \mathbb{R}\,, n \in \mathbb{N}$ and $f(t)$ is function.

Lemma 1.2: (Atanackovic, 2013) and (Podlubny, 1998)

Relation between the Caputo and Riemann-Liouville fractional derivative operator holds

$$ {}_{a}^{C}D_t^{\alpha} f(t) = {}_{a}^{RL}D_t^{\alpha}\left(f(t) - \sum_{m=0}^{n-1} \frac{(t-a)^m}{\mathrm{m!}} f^{(m)}(a) \right), t > 0, \alpha \in \mathbb{R} $$

Remark 1.5: (Podlubny, 1998)

Unlike the Riemann- Liouville fractional derivative, the Caputo fractional derivative of a constant is zero.

$$ D_t^{\alpha} C = 0 \quad , (C\ is\ a\ constant) $$

Proof:

$$ D_t^{\alpha} C = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{C^n}{(t-\tau)^{\alpha+1-n}} d\tau \ = 0. $$

For example,

Let $f(x) = C, \alpha = \frac{1}{2}, a = 0,$ then $n = 1$, so

$$ D_t^{1/2} C = \frac{1}{\Gamma(1/2)} \int_0^t (t - \tau)^{1-1+\frac{1}{2}} \ \ \frac{d}{d\tau}(C) d\tau = 0. $$

The advantages of Caputo's approach are that the derivative of the constant function under **CFDs** is zero.

5. Linear Multi-Order Fractional Differential Equation (Isah & Chang, 2015)

Definition 1.12: A linear fractional differential equation (LFDE) can be written in the form

$$D^{\alpha}y(t) = a_1 D^{\alpha_1}y(t) + \cdots + a_n D^{\alpha_n}y(t) + a_{n+1}y(t) + a_{n+2}f(t)$$

Where D^{α} is the fractional derivative operator of order α and $y(t)$ is the unknown function and $a_1, \dots, a_n$ are constant coefficients and $f(t)$ is a known function of time.

For example,

1. $D^{1.5}y(t) + y(t) = t$
2. $D^{0.5}y(t) + 2y(t) = 0$

Chapter Two

Legendre’s Polynomials

Introduction

This chapter is devoted to the study of Legendre polynomials, which are among the most prominent classical functions in mathematical analysis. Their importance lies in the fact that they serve as a canonical example of orthogonal functions over the interval $[-1,1]$, making them a fundamental theoretical tool in various numerical and analytical applications. These polynomials possess several distinctive properties, most notably the orthogonality property, which allows for accurate function representation within suitable orthogonal function spaces, in addition to normalization, which facilitates many analytical operations.

1. **Legendre's Differential Equation** (Bell, 2004) and(Sneddon & Cohen, 1956)

The Legendre equation is named after the French scientist **Adrien Marie Legendre (1752-1833),** and it is a second- order linear differential equation. Given as,

$$(1-x^2)y''-2xy'+n(n+1)y=0, \tag{2.1}$$

where n is non-negative integer.

It can also be written as follows,

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n\,(n+1)y=0$$

Clearly, the only singular points of (2.1) are $x=1$ and $x=-1$.

2. **Legendre's Polynomial** (Bell, 2004) and(Sneddon & Cohen, 1956)

They are orthogonal functions orthogonal on the interval $[-1,1]$, and are solutions to the Legendre differential equation (2.1) symbolized by $P_n(x)$, called a Legendre polynomial of order n, it is given as follows:

$$P_n(x)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{(2n-2k)!}{2^n k!\,(n-2k)!\,(n-k)}\,x^{n-2k} \tag{2.2}$$

where,

$\left[\frac{n}{2}\right]$ the largest less than or equal to $\frac{n}{2}$, that is $\left[\frac{n}{2}\right]=\begin{cases}\frac{n-1}{2}\,, & n\ odd\\ \frac{n}{2}\,, & n\ even\end{cases}$

Some properties:

When $n = 0, k = 0$ in (2.2), we have,

$$P_0(x) = (-1)^0 \frac{(2 \times 0 - 2 \times 0)!}{2^0 0!\,(0 - 2 \times 0)!\,(0 - 0)}\ \mathrm{x}^{0-2\times 0} = 1$$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2\text{-}1)$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$, etc.

Theorem 2.1: (Bell, 2004) and (Sneddon & Cohen, 1956)

(Rodrigues' Formula)

The Legendre polynomials $P_n(x)$ are defined by:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \qquad (2.3)$$

Proof:

From the binomial expansion we can write:

$$(x^2 - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (x^2)^{n-k} (-1)^k$$

$$\frac{d^n}{dx^n}(x^2 - 1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\,k!} \frac{d^n}{dx^n}(x)^{2n-2k}$$

where,

$$\frac{d^n}{dx^n}(x)^{2n-2k} = \begin{cases} \dfrac{(2n-2k)!}{(n-2k)!} x^{n-2k}, & n \le 2n - 2k \\ 0, & n > 2n - 2k \end{cases}$$

We find that

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k n!}{(n-k)!\,k!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

Thus,

$$\frac{d^n}{dx^n}(x^2-1)^n = n!\ 2^n \sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k}{2^n(n-k)!\,k!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

$$\frac{d^n}{dx^n}(x^2-1)^n = n!\ 2^n P_n(x)$$

Hence,

$$\frac{1}{n!\ 2^n}\frac{d^n}{dx^n}(x^2-1)^n = P_n(x)$$

Remark 2.1;

$$\frac{d^k}{dx^k}x^n = \begin{cases} \dfrac{n!}{(n-k)} x^{n-k}, & k \leq n \\ 0, & k \geq n \end{cases}$$

for example,

$\frac{d^7}{dx^7}x^{10} = \frac{10!}{3!}x^3.$

2.1. Generating Function of Legendre Polynomials (Bell, 2004) and(Sneddon & Cohen, 1956)

Theorem 2.2: The Legendre Polynomials can be generated as,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n\,, |x| \leq 1,\ \ |t| < 1 \tag{2.4}$$

Proof:

$$\frac{1}{\sqrt{1-2xt+t^2}} = (1-(2xt-t^2))^{-\frac{1}{2}}$$

Let $u = 2xt - t^2$

$$(1-u)^{-\frac{1}{2}} = 1+\frac{\left(-\frac{1}{2}\right)}{1!}(-u)^1+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-u)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-u)^3+\cdots$$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{(2n-1)}{2}\right)}{n!}(-u)^n+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n(1.3.5\ldots(2n-1))}{2^n n!}u^n$$

Multiply by $\frac{2^n n!}{2^n n!}$, to obtain

$$(1-u)^{-\frac{1}{2}} = \frac{2^n n!}{2^n n!}\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n\big(1.3.5\ldots(2n-1)\big)}{2^n n!}u^n$$

$$=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}u^n$$

By Substituting $u=2xt-t^2$

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}(2xt-t^2)^n$$

$$=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\ t^n(2x-t)^n$$

from the binomial expansion we find that

$$(2x-t)^n=\sum_{k=0}^{n}\binom{n}{k}(2x)^{n-k}(-t)^k$$

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\ t^n\sum_{k=0}^{n}\binom{n}{k}(2x)^{n-k}(-t)^k$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2n)!\,n!}{(n-k)!\,k!}\frac{2^{n-k}(-1)^k}{2^{2n}(n!)^2}x^{n-k}t^{n+k}$$

We find that

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(2n)!}{(n-k)!\,k!}\frac{x^{n-k}t^{n+k}}{2^{n+k}(n!)}$$

Let $\hat{n} = n + k \Longrightarrow n = \hat{n} - k$

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{\hat{n}=0}^{\infty} \sum_{k=0}^{\hat{n}} \frac{(-1)^k (2\hat{n} - 2k)!}{(\hat{n} - 2k)!\, k!} \frac{x^{\hat{n}-2k} t^{\hat{n}}}{2^{\hat{n}} (\hat{n} - k)!}$$

$$= \sum_{n=0}^{\infty} P_n(x) t^{\hat{n}}$$

2.2. Some Properties of Legendre Polynomials (Bell, 2004) and(Sneddon & Cohen, 1956)

Legendre polynomials have many Properties, the most important of which are the following:

1. $P_n(1) = 1$ (2.5)

Proof: Putting $x = 1$ in generating function (2.4)

$$\frac{1}{\sqrt{1 - 2t + t^2}} = \sum_{n=0}^{\infty} P_n(1) t^n$$

$$\sqrt{1 - 2t + t^2} = \sqrt{(1 - t)^2} = (1 - t)$$

From the Maclaurin expansion we find that

$$\frac{1}{(1 - t)} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1) t^n$$

Equal coefficients we find

$$P_n(1) = 1.$$

2. $P_n(-1) = (-1)^n$ (2.6)

Proof: Putting $x = -1$ in generating function (2.4)

$$\frac{1}{\sqrt{1 + 2t + t^2}} = \sum_{n=0}^{\infty} P_n(-1) t^n$$

$$\sqrt{1 + 2t + t^2} = \sqrt{(1 + t)^2} = (1 + t)$$

From the Maclaurin expansion we find that

$$\frac{1}{(1+t)} = \sum_{n=0}^{\infty} (-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1) t^n$$

Equal coefficients we find

$$P_n(-1) = (-1)^n$$

3. $P_n(-x) = (-1)^n P_n(x)$ (2.7)

Proof: Putting $x = -x$ in generating function (2.4)

$$\frac{1}{\sqrt{1+2t(-x)+t^2}} = \sum_{n=0}^{\infty} P_n(-x) t^n$$

It can be written in another way:

$$\frac{1}{\sqrt{1+2x(-t)+(-t)^2}} = \sum_{n=0}^{\infty} P_n(x)(-t^n)$$

$$\frac{1}{\sqrt{1+2x(-t)+(-t)^2}} = \sum_{n=0}^{\infty} P_n(x)(-1)^n t^n$$

$$\sum_{n=0}^{\infty} P_n(-x) t^n = \sum_{n=0}^{\infty} P_n(x)(-1)^n t^n$$

Equal coefficients we find

$$P_n(-x) = (-1)^n P_n(x)$$

Remark 2.2:

Form the equation (2.7) we conclude that

- $P_n(x)$ even function if n is even, that is

$$P_n(-x) = P_n(x)$$

- $P_n(x)$ odd function if n is odd, that is

$$P_n(-x) = -P_n(x)$$

2.3.Orthogonality of Legendre's Polynomials (Bell, 2004) and(Sneddon & Cohen, 1956)

Theorem 2.3: The Legendre polynomials are orthogonal on the interval $[-1,1]$ with respect to the weight function of degrees $w(x) = 1$, that is

$$\int_{-1}^{1} P_m(x) P_n(x) dx = 0, \;\; m \neq n. \tag{2.8}$$

Proof: Let $P_m(x)$ and $P_n(x)$ be Legendre polynomials of degrees n and m, respectively, where $m \neq n$. Then, we have

$$(1-x^2)P''_m(x) - 2xP'_m(x) + m(m+1)P_m(x) = 0 \qquad (2.9)$$

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0 \qquad (2.10)$$

By multiplying (2.9) in $P_n(x)$ and multiplying (2.10) in $P_m(x)$ subtract the two equations we find that

$$(1-x^2)(P_n(x)P''_m(x) - P_m(x)P''_n(x)) - 2x(P_n(x)P'_m(x) - P_m(x)P'_n(x)) = [n(n+1) - m(m+1)]P_n(x)P_m(x)$$

where,

$$P_n(x)P''_m(x) - P_m(x)P''_n(x) = \frac{d}{dx}(P_n(x)P'_m(x) - P_m(x)P'_n(x))$$

Among them, it can be written in the following form

$$\frac{d}{dx}\left[(1-x^2)\left(P_n(x)P'_m(x) - P_m(x)P'_n(x)\right)\right] = [n(n+1) - m(m+1)]P_n(x)P_m(x)$$

by conducting integration on interval [-1,1]

$$[n(n+1) - m(m+1)]\int_{-1}^{1} P_n(x)P_m(x)dx = [(1-x^2)\left(P_n(x)P'_m(x) - P_m(x)P'_n(x)\right)]_{-1}^{1}$$

$$= 0$$

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, n \neq m$$

2.4.Orthonormality Property of Legendre Polynomial (Bell, 2004) and (Sneddon & Cohen, 1956)

Theorem 2.4: The Legendre polynomials $P_n(x)$ are orthonormal on the interval $[-1,1]$ with respect to the weight function of degrees $w(x) = 1$, that is

$$\int_{-1}^{1} P^2{}_n(x)dx = \frac{2}{2n+1}. \qquad (2.8)$$

Proof: From the above equation (2.4), by squaring both sides, we find that:

by conducting integration on interval [-1,1]

$$\int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\int_{-1}^{1} P_n(x) P_m(x) dx \right] t^{n+m}$$

When $n = m$

$$-\frac{1}{2t} \ln(1 - 2xt + t^2)|_{-1}^{1} = \left[\sum_{n=0}^{\infty} \int_{-1}^{1} P^2{}_n(x) dx \right] t^{2n}$$

$$\frac{1}{t} \ln\left(\frac{1+t}{1-t}\right) = \left[\sum_{n=0}^{\infty} \int_{-1}^{1} P^2{}_n(x) dx \right] t^{2n}$$

From the Maclaurin expansion we find that

$$\sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n} = \left[\sum_{n=0}^{\infty} \int_{-1}^{1} P^2{}_n(x) dx \right] t^{2n}$$

Equating coefficients of t^{2n}, we find

$$\int_{-1}^{1} P^2{}_n(x) dx = \frac{2}{2n+1}, n = m.$$

The orthogonality and the orthonormality property of Legendre polynomial can be combined as:

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0 & ; \ m \neq n \\ \dfrac{2}{2n+1} & ; \ m = n \end{cases} \tag{2.8}$$

2.5. Recurrence Relations of the Legendre Polynomials

Theorem 2.5:

The Legendre polynomials $P_n(x)$ satisfy the following recurrence relation for $n \geq 1$:

I. $(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$ (2.11)

Proof:

Differentiating both sides of (2.4) with respect to $\boldsymbol{t}$, we get

$$-\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}}(-2x + 2t) = \sum_{n=0}^{\infty} n\, t^{n-1} P_n(x)$$

Simplifying

$$(1 - 2xt + t^2)^{-\frac{3}{2}}(x - t) = \sum_{n=0}^{\infty} n\, t^{n-1} P_n(x)$$

Multiplying both sides by $(1 - 2xt + t^2)$, we get

$$(x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} n\, t^{n-1}P_n(x)$$

using (2.4), we have

$$(x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} n\, t^{n-1}P_n(x)$$

Thus,

$$x\sum_{n=0}^{\infty} t^n P_n(x) - t\sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=0}^{\infty} n\, t^{n-1}P_n(x) - 2xt\sum_{n=0}^{\infty} n\, t^{n-1}P_n(x) + t^2\sum_{n=0}^{\infty} n\, t^{n-1}P_n(x) \tag{2.12}$$

Equating of coefficients of $\boldsymbol{t^n}$on both sides of (2.12), we get

$x\, P_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$

$\Rightarrow (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) - x\, P_n(x) + P_{n-1}(x) = 0$

We get:

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

II. $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$ (2.13)

Proof:

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Differentiating both sides of (2.4) with respect to $\boldsymbol{t}$, we get

$$(x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} n\, t^{n-1}P_n(x) \tag{2.14}$$

Again, differentiating (2.4) with respect to x, we get

$$t(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} t^n P'_n(x) \tag{2.15}$$

Multiplying both sides of (2.15) by $(x - t)$, we get:

$$t(x - t)(1 - 2xt + t^2)^{-\frac{3}{2}} = (x - t)\sum_{n=0}^{\infty} t^n P'_n(x) \qquad (2.16)$$

From (2.14) and (2.16), we get:

$$t\sum_{n=0}^{\infty} n\, t^{n-1} P_n(x) = (x - t)\sum_{n=0}^{\infty} t^n P'_n(x)$$

Hence,

$$\sum_{n=0}^{\infty} n\, t^n P_n(x) = x\sum_{n=0}^{\infty} t^n P'_n(x) - \sum_{n=0}^{\infty} t^{n+1} P'_n(x) \qquad (2.17)$$

With equality of coefficients of $\boldsymbol{t^n}$ on both sides of (2.17), we get

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

III. $(2n + 1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$ (2.18)

Proof: From recurrence formula (2.11), we get

$$(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x) \qquad (2.19)$$

Differentiating both sides of (2.19) with respect to x, we get

$$(2n + 1)xP'_n(x) + (2n + 1)P_n(x) = (n + 1)P'_{n+1}(x) + nP'_{n-1}(x) \qquad (2.20)$$

From recurrence formula (2.13) we get

$$xP'_n(x) = nP_n(x) + P'_{n-1}(x) \qquad (2.21)$$

Removing $xP'_n(x)$ from (2.20) and (2.21), we get

$$(2n + 1)[nP_n(x) + P'_{n-1}(x)] + (2n + 1)P_n(x) = (n + 1)P'_{n+1}(x) + nP'_{n-1}(x)$$

Thus,

$$(2n + 1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

Example 2.1: Express $f(x) = x^2 + x - 5$ in terms of Legendre's polynomials.

Solution: We know that

$$P_0(x) = 1 \ , P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

So, we have $x^2 = \frac{1}{3}[2P_2(x) + P_0(x)]$

Then,

$$f(x) = \frac{1}{3}[2P_2(x) + P_0(x)] + P_1(x) - 5P_0(x)]$$

Chapter Three

Wavelet Theory

Wavelet Theory (Akujuobi, 2022) and (Ray & Gupta, 2018)

The question that confuses many is what are wavelets? The answer to this question is that wavelets are positive shapes with a finite period and their average value is zero.

In this chapter, we will extensively study the different wavelet methods for obtaining numerical solutions to mathematical problems, which is a relatively new field of mathematical research. Wavelets have been used with great success in many applications, including (analyzing signals to prove the shape of the wave and dividing it, medical diagnosis, statistical analysis, …).

The main advantage of wavelets is their ability to convert differential and integral equations into linear or non-linear algebraic equations. There are many of the different methods that make it into algebraic equations, including (Haar wavelet matrix method, Daubechies wavelet matrix method, Chebyshev wavelet matrix method, Hermite wavelet matrix method, Legendre wavelet matrix method, …).

Therefore, in this chapter we will mainly apply the two- dimensional Legendre Wavelet Matrix method (LWM). The aim of this method is to examine the accuracy and efficiency of the wavelet method for solving fractional differential equations.

1. History of Wavelets

In the history of mathematics, wavelet analysis goes back to different origins, which were developed in the past twenty to thirty years independently by scientists, and several works were presented in the **1930**.

Also, before **1930** the main branch of mathematic that leads to wavelets began with **Joseph Fourier** in **1809**, which was called the **Fourier synthesis** it is written in the following form:

$$a_0 + \sum_{m=1}^{\infty} (a_m cosmx + b_m sinmx)$$

Since a_m, b_m Fourier coefficients where,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\,s\,in(mx)\,dx$$

Fourier's theory played important role in developing the ideas of mathematicians.

In **1807,** the convergence of Fourier series and orthogonality were explored. The first mention to **Haar wavelet** appeared in **1909**. On of their properties is that they have a compressed support, which means that they disappear outside a limited time interval. Unfortunately, continuous differentiation of Haar wavelet is not possible.

In **1930,** physicist **Pual Levy** used a scale-varying basis function called the Haar basis function investigated Brownian motion, a type of random signal. He found that the Haar basis function was superior to Fourier basis functions in studying the small, complex details of Brownian motion.

Also, in **1964 Calderon** studied the theory of harmonic analysis. In **1960 -1980,** mathematicians discovered atoms using **Guido Weiss and Ronald R. Coifman**, in **1980,** both **Grossman** and **Morlet** defined wavelets broadly in context of quantum physics, and the first to coin the term "wavelet" was the scientist **Jean Morlet** in **1983,** who used wavelets in seismic analysis.

We close this historical introduction by citing some of the applications which include addressing problems in computer vision, turbulence, computer graphics, seismology, image processing, signal processing.

2. **Wavelets** (Akujuobi, 2022) and (Ray & Gupta, 2018)

Wavelet Basic Function

Wavelets are functions generated from one basic function, the term "**wavelet**" was derived from the French word ***ondelette***, which means "small wave". A wavelet is oscillatory function $\phi(t) \in L^2(\Re)$ with zero mean, and is symbolized by $\phi(t)$, if it satisfies the following conditions:

I. Localization.

II. The admissibility condition

$$\int_{-\infty}^{\infty} \frac{|\phi(w)|^2}{|w|} dw < +\infty.$$

III. Smoothness.

3. Wavelet Transform (Akujuobi, 2022) and (Ray & Gupta, 2018)

In **1982**, the French scientist **Jean Morlet,** a geophysical engineer, first introduced the concept of small wavelets as a family of function and called them the "mother wavelet" it is also called "continuous wavelets" given as follows:

$$\phi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \ \phi\left(\frac{t-b}{a}\right), a.b \in \mathbb{R}, a \neq 0 \qquad (3.1)$$

where,

a is called a scaling parameter that measures the degree of compression or scale.

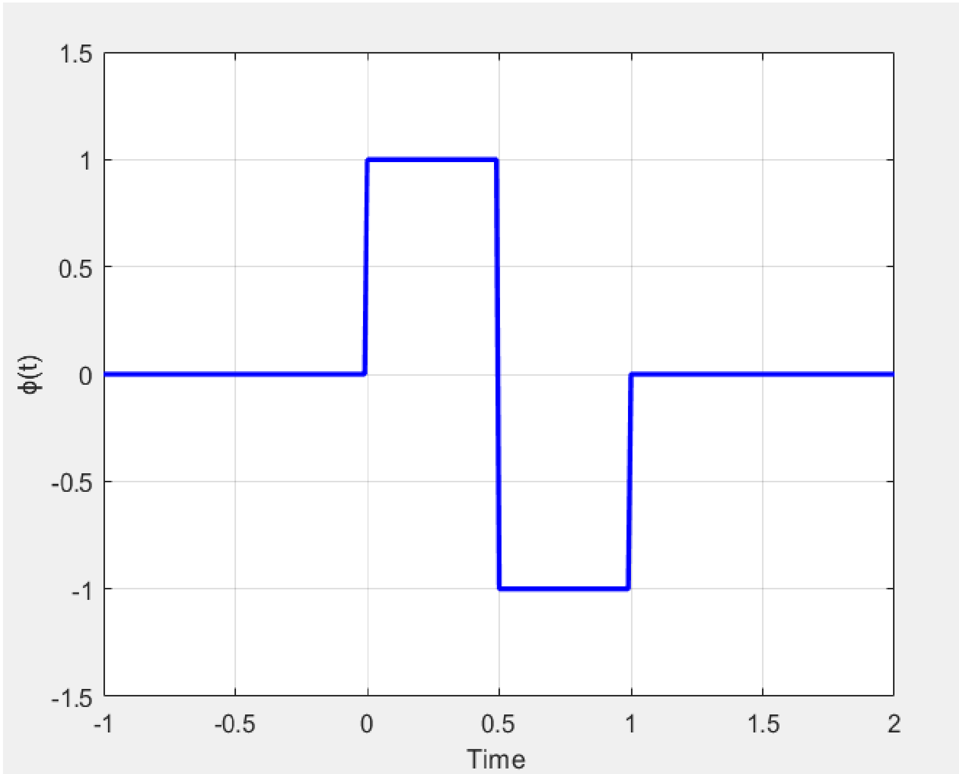
b is a translation or shifting parameter that determines the location of the wavelet.

There are many applications for wavelet transform, and one of the most important and easiest of these applications is the **Alfréd Haar wavelet**, which is most common wavelet in 1910 because it is easy to solve with and the Haar wavelet's mother wavelet function $\phi(t)$, we given:

$$\phi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq \ t < 1 \\ 0, & otherwise \end{cases}$$

Where $\phi(t)$ is the Haar wavelet orthonormal basic. Its scaling function $\phi(t)$, then we obtain

$$\phi(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & otherwise \end{cases}$$

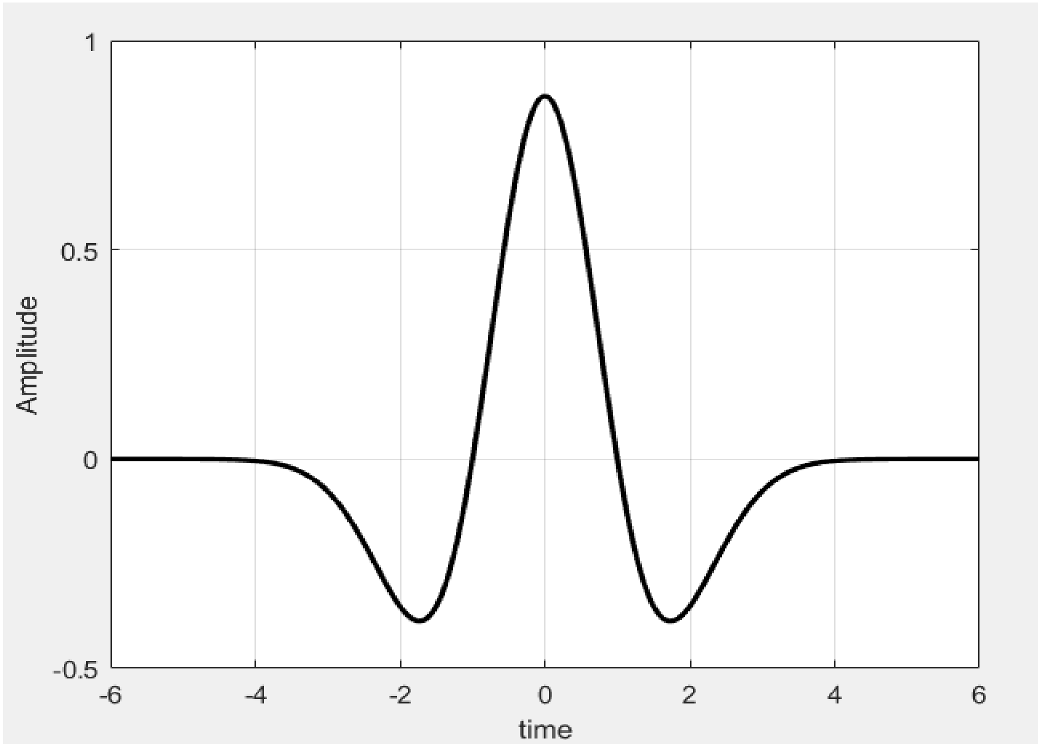


Figer 3.1: The Haar Wavelet

This means that the wavelet changes its sign suddenly $x = 0.5$.

Also, one of the most important wavelets known after Haar wavelet is **Mexican hat wavelet** it was named after **Norman. H. Ricker (1896-1980)** the Mexican hat wavelet was known as the Ricker wavelet, an American geophysicist, and it is known as the Mexican hat wavelet, especially in America, because its shape resembles a sombrero hat. They give Mexican hat wavelet as follows:

$$\phi(t) = (1 - x^2)e^{\frac{-x^2}{2}}$$



Figer 3.2: The Mexican Hat Wavelet

Remark 3.1: (Akujuobi, 2022) and (Ray & Gupta, 2018)

let $a = {a_0}^{-i}, b = nb_0{a_0}^{-i}$, $a_0 > 1, b_0 > 1$, substituting into equation (3.1) we get "Discrete wavelet" as:

$$\phi_{n,i}(t) = |a_0|^{-\frac{i}{2}}\ \phi\left({a_0}^i t - nb_0\right)$$

The set wavelet $\phi_{\ n,i}(t)$ from on orthonormal basis $L^2(\mathfrak{R})$. In particular, when a_0= 2 and b_0= 1.

4. The Basic Properties of Wavelets (Akujuobi, 2022) and (Ray & Gupta, 2018)

The most important properties of wavelets are the admissibility and the regularity conditions and these are the properties which gave wavelets their name.

The square integrable function $\phi(t)$ satisfying the admissibility condition,

$$\int_{-\infty}^{\infty} \frac{|\phi(w)|^2}{|w|} dw < +\infty \qquad (3.2)$$

can be used to first analyze and then reconstruct the signal without loss of formation. In the equation (3.2), $\phi(w)$ is the Fourier Transform of $\phi(t)$ .The admissibility condition implies that the Fourier Transform of $\phi(t)$ vanishes at zero frequency, i.e.

$$|\phi(0)|^2 = 0 \tag{3.3}$$

A zero at the zero frequency also means that the average value of the wavelet in time domain must be zero,

$$\int_{-\infty}^{\infty} \phi(t)\, dt = 0 \tag{3.4}$$

Therefore, it must be oscillatory. That is $\phi(t)$ must be a wave.

Remark 3.2:

$\phi(t)$ is n times differentiable and its derivatives are continuous and it is called (smoothness)

5. **Orthonormal and Semi-Orthogonal Wavelets**(Akujuobi, 2022) and (Ray & Gupta, 2018)

Definition 3.1: (Orthonormal Wavelet)

Orthonormal wavelet plays a major role in wavelet theory and are called $\phi \in l^2(\Re)$ Orthonormal, and the family of functions $\phi_{m,n}$ resulting from the individual function is created ϕ, which is an Orthonormal basis, and we get

$$\langle \phi_{i,j}, \phi_{m,n} \rangle = \delta_{i,m}, \delta_{j,n}; \quad i, j, m, n \in \mathbb{Z}.$$

Definition 3.2: (Semi-Orthogonal Wavelet)

Semi-Orthogonal wavelet $\phi \in l^2(\Re)$ is called Semi-Orthogonal if the family $\phi_{m,n}$ check the following condition:

$$\langle \phi_{i,j}, \phi_{m,n} \rangle = 0, i \neq m, i, j, m, n \in \mathbb{Z}.$$

6. **Shifted Legendre Polynomial and Legendre Wavelets**

(Chang et al., 2016) ,(Mohammadi & Hosseini, 2011) and (Secer et al., 2019)

6.1.Shifted Legendre polynomial

Assuming that the Legendre polynomial of degree n are defend on the interval $[-1,1]$. Then $P_n(t)$ may be following recurrence formulae

$$P_{n+1}(t) = \frac{2n+1}{n+1} t P_n(t) - \frac{n}{n+1} P_{n-1}(t), n = 1, 2, \dots \tag{3.5}$$

Definition 3.3:(Shifted Legendre Polynomial) (Chang et al., 2016), (Mohammadi & Hosseini, 2011) and (Secer et al., 2019)

The Legendre polynomial on interval [0,1] is called shifted Legendre polynomial by using the change of variable $t = 2x - 1$. Let suppose the shifted Legendre polynomial $P_n(2x-1)$ it is referred to as $P_n^*(x)$. That is,

$$P_{n+1}^*(x) = \frac{(2n+1)(2x-1)}{n+1} P_n^*(x) - \frac{n}{n+1} P_{n-1}^*(x), \;\; where\; n = 1, 2, \dots \qquad (3.6)$$

Or equivalently

$$P_n^*(x) = \sum_{\mathrm{k}=0}^{n} (-1)^{\mathrm{k}+n} \frac{(n+\mathrm{k})!}{(n-\mathrm{k})!\,(k!)^2}\; x^k \qquad (3.7)$$

Remark 3.3:

Note that $P_n^*(0) = (-1)^n$and $P_n^*(1) = 1$.

These polynomials are the orthogonality condition orthogonality property of shifted Legendre polynomials

$$\int_0^1 P_n^*(x) P_m^*(x) dx = \begin{cases} \frac{1}{2n+1}, & n = m \\ 0, & n \neq m \end{cases} \qquad (3.8)$$

Proof:

As we explained previously in section (2.8) when $n \neq m$,we can easily prove the orthogonality property.

And when $n = m$, we get

$$\int_0^1 P_n^{*2}(x) dx = \frac{1}{2n+1}$$

From the generating function, by squaring both sides, we find that:

$$\frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n^*(x)\, P_m^*(x)\, t^{n+m}$$

by conducting integration on interval [0,1]

$$\int_0^1 \frac{dx}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\int_0^1 P_n^*(x)\, P_m^*(x)\, dx \right] t^{n+m}$$

For the case $n = m$, one has, $x = 2x - 1$

$$\int_0^1 \frac{dx}{1-2t(x-\frac{1}{2})+t^2} = \left[\sum_{n=0}^{\infty}\int_0^1 P_n^{*2}(x)dx\right]t^{2n}$$

Let $u = \left(x-\frac{1}{2}\right) \Rightarrow du = dx \Rightarrow u = \frac{-1}{2}, u = \frac{1}{2}$, we get

$$-\frac{1}{4t}\ln\left(\frac{1+2t+t^2}{1-2t+t^2}\right) = \left[\sum_{n=0}^{\infty}\int_0^1 P_n^{*2}(x)dx\right]t^{2n}$$

From the Maclaurin expansion we find that,

$$\frac{1}{4t}\sum_{n=0}^{\infty}\frac{4t}{2n+1}t^{2n} = \left[\sum_{n=0}^{\infty}\int_0^1 P_n^{*2}(x)dx\right]t^{2n}$$

Equating coefficients of t^{2n}, we find

$$\int_0^1 P_n^{*2}(x)dx = \frac{1}{2n+1}$$

❖ Relation Between Shifted Legendre and Their Derivatives

Theorem 3.1: (Mohammadi & Hosseini, 2011)

Let $P_m^*(x)$ be the shifted Legendre polynomials on interval [0,1], then we get

$$P_m^{*\prime}(x) = 2\sum_{\substack{k=0\\ k+m\ odd}}^{m-1}(2k-1)P_m^*(x) \tag{3.9}$$

Proof:

Suppose that the Legendre expansion of function $v(x)$ by:

$$v(x) = \sum_{k=0}^{\infty}\hat{v}_k P_k(x) \tag{3.10}$$

Differentiating $v(x)$ in equation (3.10), we get

$$v'(x) = \sum_{k=0}^{\infty}\hat{v}_k^{(1)} P_k(x) \tag{3.11}$$

where,

$$\hat{v}_k^{(1)} = (2k+1)\sum_{\substack{p=k+1\\ p+k\ odd}}^{\infty}\hat{v}_p, \quad k \geq 0 \tag{3.12}$$

Let $v(x) = P_m(x)$ in equation (3.10), we get $\hat{v}_m = 1$ and $\hat{v}_j = 0$ for $j \neq m,$ it is

$$\hat{v}_k^{(1)} = \begin{cases} (2k+1), & m+k \; is \; odd, k \leq m-1 \\ 0, & otherwise \end{cases}$$

Substituting the value of both $v(x), \hat{v}_k^{(1)}$ in equation (3.11), we get:

$$P_m'(x) = \sum_{\substack{k=0 \\ k+m \; odd}}^{m=1} (2k+1) P_k(x) \tag{3.13}$$

By substituting $x = (2t - 1)$ in equation (3.13), we get:

$$P_m^{*\prime}(x) = 2 \sum_{\substack{k=0 \\ k+m \; odd}}^{m-1} (2k-1) P_m^*(x)$$

6.2.Legendre Wavelets (Secer et al., 2019)

As we mentioned previously, there is a large group of wavelet functions such as (Haar wavelet, Legendre wavelet, …) in this thesis, we will study the Legendre wavelet function because of its regularity and clarity.

The Legendre wavelet $\phi_{m,n}(t) = \phi(k, \hat{n}, m, t)$ have argument; k can assume any positive integer, $\hat{n} = 2n - 1$, m is order for Legendre polynomial and t is the normalized time, defined on interval [0,1] as follows:

$$\phi_{m,n}(t) = \begin{cases} 2^{\frac{k+1}{2}} \sqrt{m + \frac{1}{2}} \; P_m^*(2^k t - \hat{n}), for \; \frac{\hat{n}-1}{2} \leq t \leq \frac{\hat{n}+1}{2} \\ 0, \qquad otherwise \end{cases} \tag{2.14}$$

Where $n = 1,2, \dots, (2^k - 1)$, $k = 2,3, \dots$, $m = 0,1, \dots, M-1$ (M is fixed positive integer) and $\sqrt{m + \frac{1}{2}}$ is the orthonormality.

Remark 3.4:

Let parameter is $a = 2^{-k}$ and the translation parameter is $b = \hat{n} \, 2^{-k}$,from equation (3.1) we get:

$$\phi_{a,b}(t) = 2^{\frac{k}{2}} \phi\left(\frac{t - \hat{n} 2^{-k}}{2^{-k}}\right)$$

$$= 2^{\frac{k}{2}} \phi(2^k t - \hat{n})$$

6.3.Function Approximations (Secer et al., 2019) and (Chang et al., 2016)

Any function $y(t)$ which is square integrable and defined over [0,1] may be expended into Legendre wavelet as:

$$y(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} w_{n,m} \phi_{n,m}(t) \tag{3.15}$$

Where the coefficient $w_{n,m} = (y(t), \phi_{n,m}(t))$ is inner product.

Truncating infinite series in equation (3.15), then it can be written as:

$$y(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} w_{n,m} \phi_{n,m}(t) = W^T \, \Phi(t) \tag{3.16}$$

Were $\Phi(t)$ is matric given by:

$$\Phi = \left[\phi_{0,0}, \phi_{0,1}, \dots, \phi_{0,M}, \phi_{1,0}, \dots, \phi_{2,M}, \dots, \phi_{2^k-1,0}, \dots, \phi_{2^k-1,M}\right]^T \tag{3.17}$$

And W is matric given by:

$$W = \left[w_{0,0}, w_{0,1}, \dots, w_{0,M}, w_{1,0}, \dots, w_{2,M}, \dots, w_{2^k-1,0}, \dots, w_{2^k-1,M}\right]^T \tag{3.18}$$

Chapter Four

Numerical Solution of Fractional Differential Equations Using Shifted Legendre Wavelets

Introduction

In this chapter, we present a numerical analysis for solving fractional differential equations using Legendre wavelet matrices. We rely on transforming the differential equation into a linear system that can be easily and accurately solved. This chapter tests the proposed methodology by solving a number of numerical exercises and comparing the obtained results with the solutions. This comparison aims to evaluate the accuracy and efficiency of the approach used and to demonstrate its superiority or compatibility with other methods. We also highlight the key benefits of this method, particularly in simplifying numerical operations, stabilizing solutions, and improving computation speed, which enhances its applicability to various mathematical and physical problem.

1. Operational Matrix of Derivative and Fractional Derivative

In his section, we introduce a new Legendre wavelet operational matrix of fractional derivative.

Theorem 4.1: (Mohammadi & Hosseini, 2011)

If $\phi(t)$ is the Legendre wavelets function defined by equation (3.14) in section 3, then $\phi(t)$ may be presented as,

$$\frac{d\phi(t)}{dt} \cong D\phi(t), where\ D\ is\ 2^k(M+1) \tag{4.1}$$

$$D = \begin{bmatrix} Z & 0 & \ldots & 0 \\ 0 & Z & \ldots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & \ldots & Z \end{bmatrix} \tag{4.2}$$

Where Z is an $(M+1)(M+1)$ matrix and its elements are represented as:

$$Z_{i,j} = \begin{cases} 2^{k+1}\sqrt{(2i-1)(2j-1)}, i = 2, \ldots, (M+1), j = 1, \ldots, i-1\ and\ (i+j)\ odd \\ 0 \qquad , \quad otherwise \end{cases} \tag{4.3}$$

Proof:

We use the shifted Legendre polynomial on interval [0,1], the rth element of vector $\phi(t)$ it is written in the following from:

$$\phi_r(t) = \phi_{n,m}(t) = 2^{\frac{k+1}{2}} \sqrt{m+\frac{1}{2}}\; P_m^*(2^k t - n) x_{\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]}, i = 1, \dots 2^k(M+1) \quad (4.4)$$

Where $x_{\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]}$ is the characteristic function defined by:

$$x_{\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]} = \begin{cases} 1, & t \in \left[\frac{n}{2^k},\frac{n+1}{2^k}\right] \\ 0, & otherwise \end{cases}$$

and where $r = n(M+1) + (m+1), n = 0,1, \dots, (2^k - 1), m = 0,1,2, \dots, M.$

By differentiating equation (4.4) with respect to time t, we have

$$\frac{d\phi_r(t)}{dt} = 2^{\frac{k+1}{2}} \sqrt{m+\frac{1}{2}}\, 2^k\, P_m^{*\prime}(2^k t - n) x_{\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]} \quad (4.5)$$

This function is non-zero inside $\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]$, hence its Legendre wavelets expansion only have those elements of basis Legendre wavelets in $\phi(t)$,that is $\phi_i, i = n(M+1)+1, n(M+1)+2, \dots, n(M+1)+(M+1)$. It gives the Legendre wavelet expansion in the form that :

$$\frac{d\phi_r(t)}{dt} = \sum_{i=n(M+1)+1}^{(n+1)(M+1)} b_i\, \phi_i(t)$$

From this, we find that the first row of the matrix Z in the equation is zero because $\frac{dP_0^*(t)}{dt} = 0$. This implies $\frac{d\phi_r(t)}{dt} = 0$. After that, we substitute $P_m^{*\prime}(2^k t - n)$ from equation (3.9) into equation (4.5), and we get:

$$\frac{d\phi_r(t)}{dt} = 2^{\frac{k+1}{2}} \sqrt{m+\frac{1}{2}}\, 2^k \sum_{\substack{s=0 \\ s+m\ odd}}^{m-1} 2(2s-1) P_s^*(2^k t - n)\, x_{\left[\frac{n}{2^k},\frac{n+1}{2\ \ k}\right]} \quad (4.6)$$

Expanding this equation in Legendre wavelets basis we find that

$$\frac{d\phi_r(t)}{dt} = 2^{k+1} \sum_{\substack{j=1 \\ i+j\ odd}}^{i-1} \sqrt{(2i-1)(2j-1)}\, \phi_{n(M+1)+j(t)} \quad (4.7)$$

So, we say $Z_{i,j}$ is

$$Z_{i,j} = \begin{cases} 2^{k+1}\sqrt{(2i-1)(2j-1)}, i = 2, \dots, (M+1), j = 1, \dots, i-1\ and\ (i+j)\ odd \\ 0 \qquad , \ otherwise \end{cases}$$

For example,

We analyze row by row, **the first row**:

When $i = 0$ (there are no values for j), it is $Z_{i,j} = 0$ (for all values of j)

The second row:

when $i = 1, j = 0$ (because j must be less than i); from equation (4.3), where ($k = 1, M = 2$) we find that:

$$Z_{1,0=}2^2 \times \sqrt{(2 \times 2 - 1)(2 \times 1 - 1)} = 4\sqrt{3}$$

The rest of the values are zeros.

The third row:

when ($i = 2, j = 1$) from equation (4.3), where ($k = 1$) we find that:

$$Z_{2,1=}2^2 \times \sqrt{(2 \times 3 - 1)(2 \times 2 - 1)} = 4\sqrt{15}$$

The rest of the values are zeros.

We get the following matrix:

$$Z = \begin{bmatrix} 0 & 0 & 0 \\ 4\sqrt{3} & 0 & 0 \\ 0 & 4\sqrt{15} & 0 \end{bmatrix}$$

2. Normalized Shifted Legendre Operational Matrix of Fractional Order Derivative

Theorem 4.2: (Chang et al., 2016)

Let $\phi(x)$ be the normalized shifted Legendre polynomials, the fractional derivative of order α of the vector $\phi(x)$ can be expressed by:

$$D^{\alpha}\phi(t) = D^{(\alpha)}\phi(t) \qquad (4.8)$$

Where $D^{(\alpha)}$is the $(M+1)(M+1)$ operational matrix of fractional derivative of order α defined as:

$$
D^{(\alpha)} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 \\
\sum_{k=[\alpha]}^{[\alpha]} \theta_{[\alpha],0,k} & \sum_{k=[\alpha]}^{[\alpha]} \theta_{[\alpha],1,k} & \cdots & \sum_{k=[\alpha]}^{[\alpha]} \theta_{[\alpha],m,k} \\
\vdots & \vdots & \cdots & \vdots \\
\sum_{k=[\alpha]}^{i} \theta_{i,0,k} & \sum_{k=[\alpha]}^{i} \theta_{i,1,k} & \cdots & \sum_{k=[\alpha]}^{i} \theta_{i,m,k} \\
\vdots & \vdots & \cdots & \vdots \\
\sum_{k=[\alpha]}^{m} \theta_{m,0,k} & \sum_{k=[\alpha]}^{m} \theta_{m,1,k} & \cdots & \sum_{k=[\alpha]}^{m} \theta_{m,m,k}
\end{bmatrix}
$$

Where $\theta_{i,j,k}$ is presented by:

$$
\theta_{i,j,k} = \sqrt{2i+1}\sqrt{2j+1}\sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!\,(l+j)!}{(i+k)!\,k!\,\Gamma(k-\alpha+1)(j-l)!\,(l!)^2(k+l-\alpha+1)} \tag{4.9}
$$

Where i is the row and j is the column.

Proof:

Using $p_j(x) = \sqrt{2j+1}P_j^*(x)$, from equation (1.14) and (3.7), we get

$$
D^\alpha p_i(x) = \sqrt{2i+1}\sum_{\mathrm{k}=0}^{\mathrm{i}} (-1)^{k+\mathrm{i}} \frac{(i+k)!}{(i-k)!\,(k!)^2}\, D^\alpha x^k, i = [\alpha], \dots, m
$$

Where $D^\alpha x^k$ from Caputo's definition, it is as follows:

$$
D^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha} = \frac{k!}{\Gamma(k+1-\alpha)} x^{k-\alpha}
$$

$$
D^\alpha p_i(x) = \sqrt{2i+1}\sum_{k=0}^{[\alpha]} \frac{(-1)^{k+\mathrm{i}}(i+k)!\ x^{k-\alpha}}{(i-k)!\,(k!)\Gamma(k-\alpha+1)}, \qquad i = [\alpha], \dots, m \tag{4.10}
$$

Approximating $x^{k-\alpha}$ by $(m+1)$ term of normalized shifted Legendre polynomial, we get

$$x^{k-\alpha} \simeq \sum_{j=0}^{m} b_{k,j} p_j(x) \tag{4.11}$$

where,

$$b_{k,j} = \int_0^1 x^{k-\alpha} p_j(x) dx = \sqrt{2j+1} \sum_{l=0}^{j} (-1)^{\mathrm{j}+l} \frac{(\mathrm{j}+l)!}{(j-l)!\,(l!)^2} \int_0^1 x^{k-\alpha+l}\, dx$$

$$= \sqrt{2j+1} \sum_{l=0}^{j} \frac{(-1)^{\mathrm{j}+l}(\mathrm{j}+l)!}{(j-l)!\,(l!)^2 (k-\alpha+l+1)} \tag{4.12}$$

Utilizing equations (4.10) and (4.12), we get

$$D^{\alpha} p_i(x)$$

$$\simeq \sum_{k=[\alpha]}^{i} \sqrt{2i+1} \sqrt{2j+1} \sum_{j=0}^{m} \frac{(-1)^{k+\mathrm{i}}(i+k)!}{(i-k)!\,(k!)\Gamma(k-\alpha+1)} b_{k,j} p_j(x)$$

$$= \sum_{j=0}^{m} \left(\sum_{k=[\alpha]}^{i} \theta_{i,j,k} \right) p_j(x) \tag{4.13}$$

where $\theta_{i,j,k}$ is given in (4.9). equation (4.9) can be rewritten in vector from, we get

$$D^{\alpha} p_i(x) \simeq \left[\sum_{k=[\alpha]}^{i} \theta_{i,0,k}\,, \sum_{k=[\alpha]}^{i} \theta_{i,1,k}\,, \ldots, \sum_{k=[\alpha]}^{i} \theta_{i,m,k} \right] \Phi(x) \tag{4.14}$$

3. Solving Linear Fractional Differential Equations (Secer et al., 2019)

We will apply the Legendre wavelet operator matrix **(LWOM)** to the fractional derivative to solve linear fractional differential equations given as,

$$D^{\alpha} y(t) = u_0 D^{\mu_0} y(t) + \cdots + u_{k-1}(t) D^{\mu_{k-1}} y(t) + u_k(t) y(t) + f(t) \tag{4.15}$$

with initial conditions

$$y(t_0) = y_0, \frac{dy}{dt}(t_0) = y_1, \frac{d^2 y}{dt^2}(t_0) = y_2, \ldots,, \frac{d^{n-1} y}{dt^{n-1}}(t_0) = y_{n-1} \tag{4.16}$$

Where D^{α}denotes the Caputo fractional derivative of order α and $u_0(t), u_1(t), \ldots, u_k(t)$ are functions and $0 < \mu_0 < \mu_1 < \cdots < \mu_{k-1} < \alpha$.

First approximate $y(t), f(t)$ and $u_0(t), u_1(t), \ldots, u_k(t)$ by the Legendre wavelets, we have

$$y(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} w_{n,m}\phi_{n,m} = W^T\Phi(t) \tag{4.17}$$

Where $W \; and \; \Phi$ are $2^k(M+1) \times 1$ matrices given by

$$W = \left[w_{0,0}, w_{0,1}, \ldots, w_{0,M}, w_{1,0}, w_{1,1}, \ldots, w_{1,M}, \ldots, w_{(2^k-1),0}, w_{(2^k-1),1}, \ldots, w_{(2^k-1),M}\right]^T$$

$$\Phi(t) = \left[\phi_{0,0}, \phi_{0,1}, \ldots, \phi_{0,M}, \phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,M}, \ldots, \phi_{(2^k-1),0}, \phi_{(2^k-1),1}, \ldots, \phi_{(2^k-1),M}\right]^T$$

We find $\phi_{0,0}$ the is found using the equation (3.10), we get:

Let $k = 0, n = 0 \; and \; m = 0$,we have

$$\phi_{0,0} = 2^{\frac{0+1}{2}}\sqrt{0+\frac{1}{2}}P_0^*(2^0t-0) = 2^{\frac{1}{2}}\sqrt{2} \times 1 = 1$$

Let $k = 0, n = 0 \; and \; m = 1$,we have

$$\phi_{0,1} = 2^{\frac{0+1}{2}}\sqrt{1+\frac{1}{2}}P_1^*(2^0t-0) = 2^{\frac{1}{2}} \times \sqrt{\frac{3}{2}} \times t = \sqrt{3}t$$

$$\phi_{1,1} = 2^{\frac{0+1}{2}}\sqrt{1+\frac{1}{2}}P_1^*(2^0t-1) = P_1^*(2^0t-1) \times 2^{\frac{1}{2}} \times \sqrt{\frac{3}{2}} = \sqrt{3}(t-1)$$

And similarly for other functions $\phi_{n,m}$.

Kowing that for $k = 0, n = 0$, we have

For $m = 0$, one has.

$P_0^*(2^0t-0) = P_0^*(t) = 1.$

For $m = 1$, one has.

$P_1^*(2^0t-0) = P_1^*(t) = t.$

When $k = 0, n = 1$ and $m = 1$

$P_1^*(2^0t-1) = P_1^*(t-1) = t-1$

Also,

$$f(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} f_{n,m}\phi_{n,m} = F^T \Phi(t) \quad (4.18)$$

Where F and Φ are $2^k(M+1) \times 1$ vectors given by

$$\Phi(t) = \left[\phi_{0,0}, \phi_{0,1}, \dots, \phi_{0,M}, \phi_{1,0}, \phi_{1,1}, \dots, \phi_{1,M}, \dots, \phi_{(2^k-1),0}, \phi_{(2^k-1),1}, \dots, \phi_{(2^k-1),M}\right]^T$$

$$F = \left[f_{0,0}, f_{0,1}, \dots, f_{0,M}, f_{1,0}, f_{1,1}, \dots, f_{1,M}, \dots, f_{(2^k-1),0}, f_{(2^k-1),1}, \dots, f_{(2^k-1),M}\right]^T$$

And

$$u_i(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} u_{in,m}\phi_{n,m} = U_i^T \Phi(t),$$

Where $y(t)$ is unknown function , $f(t)$ is source function or external force and F and $U_i(i = 0,1, \dots k)$ are known vectors, $\Phi(t)$is the vector given in equation (3.10) and W is an unknown vector. We use equations (4.8) and (4.17), to obtain

$$D^\alpha y(t) \cong W^T D^\alpha \Phi(t) \cong W^T D^{(\alpha)} \Phi(t) \quad (4.19)$$

Where, $D^{\mu_i} y(t) = D^{\mu_0} y(t), \dots, D^{\mu_{k-1}} y(t)$ we get by

$$D^{\mu_i} y(t) \cong W^T D^{\mu_i} \Phi(t) \cong W^T D^{(\mu_i)} \Phi(t), i = 0,1, \dots, (k-1) \quad (4.20)$$

Substituting equations. (4.17) and (4.20) into equation (4.15), we find that

$$W^T D^{(\alpha)} \Phi(t) = [U_0^T \Phi(t)][W^T D^{\mu_0} \Phi(t)] + \cdots + [U_{k-1}^T \Phi(t)][W^T D^{\mu_{k-1}} \Phi(t)] + [U_k^T \Phi(t)][W^T D^{\mu_k} \Phi(t)] + [U_{k+1}^T \Phi(t)][F^T \Phi(t)]$$

The residual $R(t)$ can be expressed,

$$\begin{aligned} R(t) \cong W^T D^{(\alpha)} \Phi(t) - \left[\left(U_0^T(t)\right)\left(W^T D^{\mu_0} \Phi(t)\right)\right] - \cdots \\ - \left[\left(U_{k-1}^T \Phi(t)\right)\left(W^T D^{\mu_{k-1}} \Phi(t)\right)\right] \\ - [(U_k^T \Phi(t))(W^T D^{\mu_k} \Phi(t))] - [(U_{k+1}^T \Phi(t))(F^T \Phi(t))] \end{aligned} \quad (4.21)$$

Where $R(t)$ it is the residual and expresses the error resulting from the approximation.

Thus, we obtain $2^k(M+1) - n$ linear equations by using the following condition,

$$\langle R(t), \phi_s(t) \rangle = \int_0^1 \phi_s(t) R(t) dt = 0, s = 0, \dots, 2^k(M+1) - n \quad (4.22)$$

If we replace the equation (4.17) in equation (4.16), we find that

$$y(t_0) = W^T \phi(t_0) = y_0, \frac{dy}{dt}(t_0) = W^T D\phi(t_0) = y_1$$

$$\frac{d^2y}{dt^2}(t_0) = W^T D^2\phi(t_0) = y_2, \dots, \frac{d^ny}{dt^n}(t_0) = W^T D^{(n)}\phi(t_0) = y_n \qquad (4.23)$$

$y(t)$ can be found from equation (4.16) and from equation (4.22) and (4.23), these linear equations can be solved for unknown coefficients of the vector W.

4. Numerical Results and Discussion

Example 1:(Secer et al., 2019)

consider the following linear initial values problem that consists the fractional differential equations

$$4(t+1)D^{\frac{5}{2}}y(t) + 4D^{\frac{3}{2}}y(t) + \frac{1}{\sqrt{t+1}}y(t) = \sqrt{t} + \sqrt{\pi} \qquad (4.24)$$

And the initial conditions,

$$y(0) = \sqrt{\pi}, \frac{dx}{dt}(0) = \frac{\sqrt{\pi}}{2}, y(1) = \sqrt{2\pi}$$

Where the exact solution of this problem is given as

$$y(t) = \sqrt{\pi(t+1)}$$

Solution:

First, the matrices $D, D^{3/2} and\, D^{5/2}$ are derived mathematically, and subsequently incorporated into the operational framework to determine the numerical solution.

When $M = 3 and\; k = 0$, We find D the matrix D is found by the theorem in (4.1)

Where order $\alpha = 1$. Hence the first row is zero because $\alpha \geq 1$ and $Z_{i,j} = 2^{k+1}\sqrt{(2i-1)(2j-1)}$, i is row and j is column if $Z_{1,1} = zero, i > j$.

When $k = 0$ find that:

$$Z_{2,1} = 2^1\sqrt{(2\times 2-1)(2\times 1-1)} = 2\sqrt{3\times 1} = 2\sqrt{3}$$

$$Z_{3,2} = 2^1\sqrt{(2\times 3-1)(2\times 2-1)} = 2\sqrt{5\times 3} = 2\sqrt{15}$$

$$Z_{4.3} = 2^1\sqrt{(2\times 4-1)(2\times 3-1)} = 2\sqrt{7\times 5} = 2\sqrt{35}$$

$$Z_{5,4} = 2^1\sqrt{(2\times 5-1)(2\times 4-1)} = 2\sqrt{9\times 7} = 2\sqrt{63}$$

“And the rest of the elements of the matrix are zeros, as demonstrated the theorem in (4.1)”, we get:

$$\boldsymbol{D} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{2\sqrt{3}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2\sqrt{15}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2\sqrt{35}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2\sqrt{63}} & \mathbf{0} \end{bmatrix}$$

Based on Theorem (4.2), we proceed to derive the fractional matrix of order 3/2.

$$D^{(\alpha)} = \begin{bmatrix} 0 & 0 & \cdot\ \cdot\ \cdot & 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 & \cdot\ \cdot\ \cdot & 0 \\ \sum_{k=[\alpha]}^{[\alpha]} \theta_{[\alpha],0,k} & \sum_{k=[\alpha]}^{[\alpha]} \theta_{[\alpha],1,k} & \cdot\ \cdot\ \cdot & \sum_{k=[\alpha]}^{[\alpha]} \theta_{[\alpha],m,k} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & ; \\ \sum_{k=[\alpha]}^{i} \theta_{i,0,k} & \sum_{k=[\alpha]}^{i} \theta_{i,1,k} & \cdot\ \cdot\ \cdot & \sum_{k=[\alpha]}^{i} \theta_{i,m,k} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \sum_{k=[\alpha]}^{m} \theta_{m,0,k} & \sum_{k=[\alpha]}^{m} \theta_{m,1,k} & \cdot\ \cdot\ \cdot & \sum_{k=[\alpha]}^{m} \theta_{m,m,k} \end{bmatrix}$$

Where $\theta_{i,j,k}$ is presented by:

$$\theta_{i,j,k} = \sqrt{2i+1}\sqrt{2j+1}\sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!\,(l+j)!}{(i-k)!\,k!\,\Gamma(k-\alpha+1)(j-l)!\,(l!)^2(k+l-\alpha+1)}$$

Where i row, j column and $k = \alpha$ is integer, α order Caputo derivative. When $M = 3$ and $D^{3/2}$ we find:

“The first rows of the matrix are zeros since the order of the matrix is 2; therefore, both the First and second rows are zero”

Now, when $i = 2, j = 0, l = 0, k = 2\ and\ \alpha = 3/2$, we get

$\theta_{2,0,2}$

$$= \sqrt{2\times 2+1}\sqrt{2\times 0+1}\sum_{l=0}^{0} \frac{(-1)^{2+0+2+0}(2+2)!\,(0+0)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(0-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= \sqrt{5}\frac{(-1)^4\times 4!}{2!\times\Gamma\left(\frac{3}{2}\right)\times\left(\frac{3}{2}\right)} = \frac{16\sqrt{5}}{\sqrt{\pi}}$$

Now, when $i = 2, j = 1, k = 2 \; and \; \alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{2,1,2} = \sqrt{2 \times 2 + 1}\sqrt{2 \times 1 + 1}\frac{(-1)^{2+1+2+0}(2+2)!\,(1+0)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(1-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= -\sqrt{5}\sqrt{3}\frac{4! \times 1!}{2! \times \Gamma\left(\frac{3}{2}\right) \times \left(\frac{3}{2}\right)} = \frac{-16\sqrt{5}\sqrt{3}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 2 + 1}\sqrt{2 \times 1 + 1}\frac{(-1)^{2+1+2+1}(2+2)!\,(1+1)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(1-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= \sqrt{5}\sqrt{3}\frac{4! \times 2!}{2! \times \Gamma\left(\frac{3}{2}\right) \times \left(\frac{5}{2}\right)} = \frac{96\sqrt{5}\sqrt{3}}{5\sqrt{\pi}}$$

From $l = 0$ and $l = 1$,we obtain,

$$\frac{5}{5} \times \frac{-16\sqrt{5}\sqrt{3}}{\sqrt{\pi}} + \frac{96\sqrt{5}\sqrt{3}}{5\sqrt{\pi}} = \frac{16\sqrt{5}\sqrt{3}}{5\sqrt{\pi}}$$

Now, when $i = 2, j = 2, k = 2 \; and \; \alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{2,2,2} = \sqrt{2 \times 2 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{2+2+2+0}(2+2)!\,(2+0)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= \sqrt{5}\sqrt{5}\frac{4! \times 2!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 2! \times \left(\frac{3}{2}\right)} = \frac{80}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 2 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{2+2+2+1}(2+2)!\,(2+1)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= -\sqrt{5}\sqrt{5}\frac{4! \times 3!}{2! \times \Gamma\left(\frac{3}{2}\right) \times \left(\frac{5}{2}\right)} = \frac{-288}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 2 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{2+2+2+2}(2+2)!\,(2+2)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-2)!\,(2!)^2\left(2+2-\frac{3}{2}+1\right)}$$

$$= \sqrt{5}\sqrt{5}\frac{4! \times 4!}{2! \times \Gamma\left(\frac{3}{2}\right) \times (2!)^2 \times \left(\frac{7}{2}\right)} = \frac{1440}{7\sqrt{\pi}}$$

From $l = 0$, $l = 1$ and $l = 2$, we obtain,

$$\frac{1440}{7\sqrt{\pi}} + \frac{7}{7} \times \frac{80}{\sqrt{\pi}} - \frac{7}{7} \times \frac{-288}{\sqrt{\pi}} = \frac{-16}{7\sqrt{\pi}}$$

Now, when $i = 2, j = 3, k = 2\ and\ \alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{2,3,2} = \sqrt{2 \times 2 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{2+3+2+0}(2+2)!\,(3+0)!}{(2-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(3-0)!\,(0!)^2\left(2 + 0 - \frac{3}{2} + 1\right)}$$

$$= -\sqrt{5}\sqrt{7}\frac{4! \times 3!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 3! \times \left(\frac{3}{2}\right)} = \frac{-16\sqrt{5}\sqrt{7}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 2 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{2+3+2+1}(2+2)!\,(3+1)!}{(2-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(3-1)!\,(1!)^2\left(2 + 1 - \frac{3}{2} + 1\right)}$$

$$= \sqrt{5}\sqrt{7}\frac{4! \times 4!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 2! \times \left(\frac{5}{2}\right)} = \frac{576\sqrt{5}\sqrt{7}}{5\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 2 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{2+3+2+2}(2+2)!\,(3+2)!}{(2-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(3-2)!\,(2!)^2\left(2 + 2 - \frac{3}{2} + 1\right)}$$

$$= -\sqrt{5}\sqrt{7}\frac{4! \times 5!}{2! \times \Gamma\left(\frac{3}{2}\right) \times (2!)^2 \times \left(\frac{7}{2}\right)} = \frac{-1440\sqrt{5}\sqrt{7}}{7\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$\sqrt{2 \times 2 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{2+3+2+3}(2+2)!\,(3+3)!}{(2-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(3-3)!\,(3!)^2\left(2 + 3 - \frac{3}{2} + 1\right)}$$

$$= \sqrt{5}\sqrt{7}\frac{4! \times 6!}{2! \times \Gamma\left(\frac{3}{2}\right) \times (3!)^2 \times \left(\frac{9}{2}\right)} = \frac{320\sqrt{5}\sqrt{7}}{3\sqrt{\pi}}$$

From $l = 0$, $l = 1$, $l = 2$ and $l = 3$, we obtain,

$$\frac{\sqrt{5}\sqrt{7}}{\sqrt{\pi}}\left[\frac{105}{105}\times -16+\frac{7\times 3}{7\times 3}\times\frac{576}{5}-\times\frac{3\times 5}{3\times 5}\times\frac{1440}{7}+\frac{7\times 5}{7\times 5}\times\frac{320}{3}\right]=\frac{16\sqrt{5}\sqrt{7}}{105\sqrt{\pi}}$$

Now, when $i = 2, j = 4, k = 2\ and\ \alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{24,2}=\sqrt{2\times 2+1}\sqrt{2\times 4+1}\frac{(-1)^{2+4+2+0}(2+2)!\,(4+0)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$=\frac{16\sqrt{5}\sqrt{9}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$=\sqrt{2\times 2+1}\sqrt{2\times 4+1}\frac{(-1)^{2+4+2+1}(2+2)!\,(4+1)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$=\frac{-192\sqrt{5}\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$=\sqrt{2\times 2+1}\sqrt{2\times 4+1}\frac{(-1)^{2+4+2+2}(2+2)!\,(4+2)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-2)!\,(2!)^2\left(2+2-\frac{3}{2}+1\right)}$$

$$=\frac{4320\sqrt{5}\sqrt{9}}{7\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$=\sqrt{2\times 2+1}\sqrt{2\times 4+1}\frac{(-1)^{2+4+2+3}(2+2)!\,(4+3)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-3)!\,(3!)^2\left(2+3-\frac{3}{2}+1\right)}$$

$$=\frac{-2240\sqrt{5}\sqrt{9}}{\sqrt{\pi}}$$

Fifth when $l = 4$, we find

$$=\sqrt{2\times 2+1}\sqrt{2\times 4+1}\frac{(-1)^{2+4+2+4}(2+2)!\,(4+4)!}{(2-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-4)!\,(4!)^2\left(2+4-\frac{3}{2}+1\right)}$$

$$=\frac{3360\sqrt{5}\sqrt{9}}{11\sqrt{\pi}}$$

From $l = 0$, $l = 1$, $l = 2, l = 3$ and $l = 4$, we obtain,

$$\frac{16\sqrt{5}\sqrt{9}}{\sqrt{\pi}} - \frac{192\sqrt{5}\sqrt{9}}{\sqrt{\pi}} + \frac{4320\sqrt{5}\sqrt{9}}{7\sqrt{\pi}} - \frac{2240\sqrt{5}\sqrt{9}}{\sqrt{\pi}} + \frac{3360\sqrt{5}\sqrt{9}}{11\sqrt{\pi}} = \frac{26\sqrt{5}\sqrt{9}}{231\sqrt{\pi}}$$

Now, when $i = 3, j = 0, k = 2, \alpha = 3/2$ and $l = 0$, we get

$$\theta_{3,0,2} = \sqrt{2\times 3+1}\sqrt{2\times 0+1}\frac{(-1)^{3+0+2+0}(3+2)!\,(0+0)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(0-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= -\sqrt{7}\frac{5!}{2!\times\Gamma\left(\frac{3}{2}\right)\times\left(\frac{3}{2}\right)} = \frac{-80\sqrt{7}}{\sqrt{\pi}}$$

Now, when $i = 3, j = 0, k = 2\ and\ \alpha = 3/2$, we get:

First when $l = 0$, we find

$$\theta_{3,1,2} = \sqrt{2\times 3+1}\sqrt{2\times 1+1}\frac{(-1)^{3+1+2+0}(3+2)!\,(1+0)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(1-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{3}\frac{5!}{2!\times\Gamma\left(\frac{3}{2}\right)\times\left(\frac{3}{2}\right)} = \frac{80\sqrt{7}\sqrt{3}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2\times 3+1}\sqrt{2\times 1+1}\frac{(-1)^{3+1+2+1}(3+2)!\,(1+1)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(1-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{3}\frac{5!\times 2!}{2!\times\Gamma\left(\frac{3}{2}\right)\times\left(\frac{5}{2}\right)} = \frac{-96\sqrt{7}\sqrt{3}}{\sqrt{\pi}}$$

From $l = 0$ and $l = 1$, we obtain,

$$\frac{80\sqrt{7}\sqrt{3}}{\sqrt{\pi}} - \frac{96\sqrt{7}\sqrt{3}}{\sqrt{\pi}} = \frac{-16\sqrt{7}\sqrt{3}}{\sqrt{\pi}}$$

Now, when $i = 3, j = 2, k = 2\ and\ \alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{3,2,2} = \sqrt{2\times 3+1}\sqrt{2\times 2+1}\frac{(-1)^{3+2+2+0}(3+2)!\,(2+0)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{5}\frac{5!\times 2!}{2!\times\Gamma\left(\frac{3}{2}\right)\times 2!\times\left(\frac{3}{2}\right)} = \frac{-80\sqrt{7}\sqrt{5}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{3+2+2+1}(3+2)!\,(2+1)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{5}\frac{5! \times 3!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 2! \times \left(\frac{5}{2}\right)} = \frac{288\sqrt{7}\sqrt{5}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{3+2+2+2}(3+2)!\,(2+2)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-2)!\,(2!)^2\left(2+2-\frac{3}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{5}\frac{5! \times 4!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 4 \times \left(\frac{7}{2}\right)} = \frac{-1440\sqrt{7}\sqrt{5}}{7\sqrt{\pi}}$$

From $l = 0$, $l = 1$ and $l = 2$, we obtain,

$$\frac{7}{7} \times \frac{-80\sqrt{7}\sqrt{5}}{\sqrt{\pi}} + \frac{7}{7} \times \frac{288\sqrt{7}\sqrt{5}}{\sqrt{\pi}} - \frac{1440\sqrt{7}\sqrt{5}}{7\sqrt{\pi}} = \frac{16\sqrt{7}\sqrt{5}}{7\sqrt{\pi}}$$

Now, when $i = 3, j = 3, k = 2\ and\ \alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{3,3,2} = \sqrt{2 \times 3 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{3+3+2+0}(3+2)!\,(3+0)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(3-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{7}\frac{5! \times 3!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 3! \times \left(\frac{3}{2}\right)} = \frac{560}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{3+3+2+1}(3+2)!\,(3+1)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(3-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{7}\frac{5! \times 4!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 2! \times \left(\frac{5}{2}\right)} = \frac{-4032}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{3+3+2+2}(3+2)!\,(3+2)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(3-2)!\,(2!)^2\left(2+2-\frac{3}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{7}\frac{5! \times 5!}{2! \times \Gamma\left(\frac{3}{2}\right) \times 4 \times \left(\frac{7}{2}\right)} = \frac{7200}{\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{3+3+2+3}(3+2)!\,(3+3)!}{(3-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(3-3)!\,(3!)^2\left(2 + 3 - \frac{3}{2} + 1\right)}$$

$$= -\sqrt{7}\sqrt{7}\frac{5! \times 6!}{2! \times \Gamma\left(\frac{3}{2}\right) \times (3!)^2 \times \left(\frac{9}{2}\right)} = \frac{-11200}{3\sqrt{\pi}}$$

From $l = 0$, $l = 1, l = 2$ and $l = 3$, we obtain,

$$\frac{3}{3} \times \frac{560}{\sqrt{\pi}} - \frac{3}{3} \times \frac{4032}{\sqrt{\pi}} + \frac{3}{3} \times \frac{7200}{\sqrt{\pi}} - \frac{11200}{3\sqrt{\pi}} = \frac{-16}{3\sqrt{\pi}}$$

$$\boldsymbol{D^{3/2}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\mathbf{16\sqrt{5}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{16\sqrt{5}\sqrt{3}}}{\mathbf{5}\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{-16}}{\mathbf{7}\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{16\sqrt{5}\sqrt{7}}}{\mathbf{105}\sqrt{\boldsymbol{\pi}}} \\ \frac{\mathbf{-80\sqrt{7}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{-16\sqrt{7}\sqrt{3}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{16\sqrt{7}}\sqrt{5}}{\mathbf{7}\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{-16}}{\mathbf{3}\sqrt{\boldsymbol{\pi}}} \end{bmatrix}$$

Now, when the matrix is $\boldsymbol{D^{5/2}}$then $k = 3$ and the rows is equal to zeros.

Now, when $i = 3, j = 0, k = 3, l = 0$ *and* $\alpha = 5/2$, we get

$$\theta_{3,0,3} = \sqrt{2 \times 3 + 1}\sqrt{2 \times 0 + 1}\frac{(-1)^{3+0+3+0}(3+3)!\,(0+0)!}{(3-3)!\,3!\,\Gamma\left(3 - \frac{5}{2} + 1\right)(0-0)!\,(0!)^2\left(3 + 0 - \frac{5}{2} + 1\right)}$$

$$= \sqrt{7}\frac{6!}{3! \times \Gamma\left(\frac{3}{2}\right) \times \left(\frac{3}{2}\right)} = \frac{160\sqrt{7}}{\sqrt{\pi}}$$

Now, when $i = 3, j = 1, k = 3$ *and* $\alpha = 5/2$, we get:

First when $l = 0$, we find

$$\theta_{3,1,3} = \sqrt{2 \times 3 + 1}\sqrt{2 \times 1 + 1}\frac{(-1)^{3+1+3+0}(3+3)!\,(1+0)!}{(3-3)!\,3!\,\Gamma\left(3 - \frac{5}{2} + 1\right)(1-0)!\,(0!)^2\left(3 + 0 - \frac{5}{2} + 1\right)}$$

$$= -\sqrt{7}\sqrt{3}\frac{6!}{3! \times \Gamma\left(\frac{3}{2}\right) \times \left(\frac{3}{2}\right)} = \frac{-160\sqrt{7}\sqrt{3}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 1 + 1}\frac{(-1)^{3+1+3+1}(3+3)!\,(1+1)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(1-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{3}\frac{6! \times 2!}{3! \times \Gamma\left(\frac{3}{2}\right) \times \left(\frac{5}{2}\right)} = \frac{192\sqrt{7}\sqrt{3}}{\sqrt{\pi}}$$

From $l = 0$and $l = 1$, we obtain,

$$\frac{192\sqrt{7}\sqrt{3}}{\sqrt{\pi}} - \frac{160\sqrt{7}\sqrt{3}}{\sqrt{\pi}} = \frac{32\sqrt{7}\sqrt{3}}{\sqrt{\pi}}$$

Now, when $i = 3, j = 2, k = 3\ and\ \alpha = 5/2$, we get

First when $l = 0$, we find

$$\theta_{3,2,3} = \sqrt{2 \times 3 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{3+2+3+0}(3+3)!\,(2+0)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(2-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{5}\frac{6! \times 2!}{3! \times \Gamma\left(\frac{3}{2}\right) \times (2)! \times \left(\frac{3}{2}\right)} = \frac{160\sqrt{7}\sqrt{5}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{3+2+3+1}(3+3)!\,(2+1)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(2-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{5}\frac{6! \times 3!}{3! \times \Gamma\left(\frac{3}{2}\right) \times \left(\frac{5}{2}\right)} = \frac{-576\sqrt{7}\sqrt{5}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{3+2+3+2}(3+3)!\,(2+2)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(2-2)!\,(2!)^2\left(3+2-\frac{5}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{5}\frac{6! \times 4!}{3! \times \Gamma\left(\frac{3}{2}\right) \times (2!)^2 \times \left(\frac{7}{2}\right)} = \frac{2880\sqrt{7}\sqrt{5}}{7\sqrt{\pi}}$$

From $l = 0, l = 1$ and $l = 2$, we obtain,

$$\frac{7}{7} \times \frac{160\sqrt{7}\sqrt{5}}{\sqrt{\pi}} - \frac{7}{7} \times \frac{576\sqrt{7}\sqrt{5}}{\sqrt{\pi}} + \frac{2880\sqrt{7}\sqrt{5}}{7\sqrt{\pi}} = \frac{-32\sqrt{7}\sqrt{5}}{7\sqrt{\pi}}$$

Now, when $i = 3, j = 3, k = 3\ and\ \alpha = 5/2$, we get

First when $l = 0$, we find

Now, when $i = 3, j = 3, k = 3\ and\ \alpha = 5/2$, we get

First when $l = 0$, we find

$$\theta_{3,3,3} = \sqrt{2\times 3+1}\sqrt{2\times 3+1}\frac{(-1)^{3+3+3+0}(3+3)!\,(3+0)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{7}\frac{6!\times 3!}{3!\times\Gamma\left(\frac{3}{2}\right)\times(3)!\times\left(\frac{3}{2}\right)} = \frac{-1120}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2\times 3+1}\sqrt{2\times 3+1}\frac{(-1)^{3+3+3+1}(3+3)!\,(3+1)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{7}\frac{6!\times 4!}{3!\times\Gamma\left(\frac{3}{2}\right)\times(2)!\times\left(\frac{5}{2}\right)} = \frac{8064}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2\times 3+1}\sqrt{2\times 3+1}\frac{(-1)^{3+3+3+2}(3+3)!\,(3+2)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-2)!\,(2!)^2\left(3+2-\frac{5}{2}+1\right)}$$

$$= -\sqrt{7}\sqrt{7}\frac{6!\times 5!}{3!\times\Gamma\left(\frac{3}{2}\right)\times(2!)^2\times\left(\frac{7}{2}\right)} = \frac{-14400}{\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2\times 3+1}\sqrt{2\times 3+1}\frac{(-1)^{3+3+3+3}(3+3)!\,(3+3)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-3)!\,(3!)^2\left(3+3-\frac{5}{2}+1\right)}$$

$$= \sqrt{7}\sqrt{7}\frac{6!\times 6!}{3!\times\Gamma\left(\frac{3}{2}\right)\times(3!)^2\times\left(\frac{9}{2}\right)} = \frac{22400}{3\sqrt{\pi}}$$

From $l = 0, l = 1, l = 2$ and $l = 3$, we obtain,

$$\frac{3}{3}\times\frac{-1120}{\sqrt{\pi}} + \frac{3}{3}\times\frac{8064}{\sqrt{\pi}} - \frac{3}{3}\times\frac{14400}{\sqrt{\pi}} + \frac{22400}{3\sqrt{\pi}} = \frac{32}{3\sqrt{\pi}}$$

Now, when $i = 4, j = 0, k = 3, l = 0\ and\ \alpha = 5/2$, we get

$$\theta_{4,0,3} = \sqrt{2\times 4+1}\sqrt{2\times 0+1}\frac{(-1)^{4+0+3+0}(4+3)!\,(0+0)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(0-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \frac{-3360}{\sqrt{\pi}}$$

$$\boldsymbol{D^{5/2}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\mathbf{160}\sqrt{7}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{32}\sqrt{\mathbf{7}}\sqrt{\mathbf{3}}}{\sqrt{\boldsymbol{\pi}}} & \frac{-\mathbf{32}\sqrt{\mathbf{7}}\sqrt{\mathbf{5}}}{\mathbf{7}\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{32}}{\mathbf{3}\sqrt{\boldsymbol{\pi}}} \end{bmatrix}$$

Table 4.1: Comparison Between the Exact Solution $y(t)$ and the Approximate Solution $y_{LWOMM}(t)$ for $M = 3$

t	**Exact Solution**	**Approximate Solution**	**Absolute error**
0.0	1.77245385	1.77245337	4.8035e-07
0.1	1.85896528	1.85955549	0.0005902073
0.2	1.94162591	1.94264834	0.0010224288
0.3	2.02090832	2.02209502	0.0011866994
0.4	2.09719568	2.09825863	0.0010629481
0.5	2.17080376	2.17150225	0.0006984865
0.6	2.24199649	2.24218899	0.0001925007
0.7	2.31099708	2.31068193	0.0003151486
0.8	2.37799638	2.37734418	0.0006521962
0.9	2.44315903	2.44253883	0.0006201987
1.0	2.50662827	2.50662897	6.9768e-07

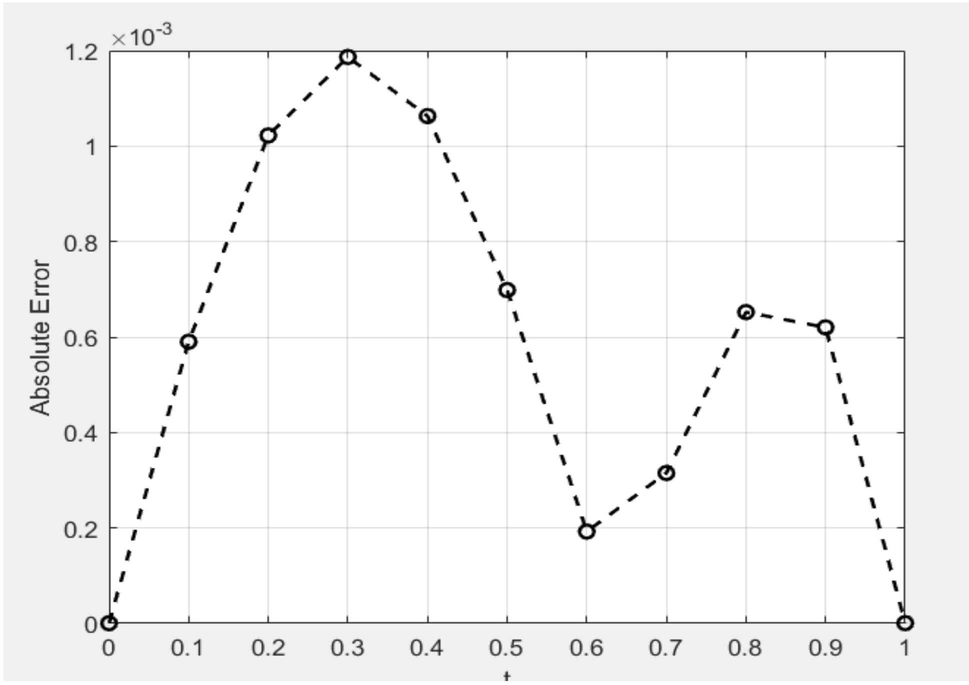


Figure 4.1: Absolute Error Between Exact and Approximate Solutions for Equation (4.24) at $M = 3$

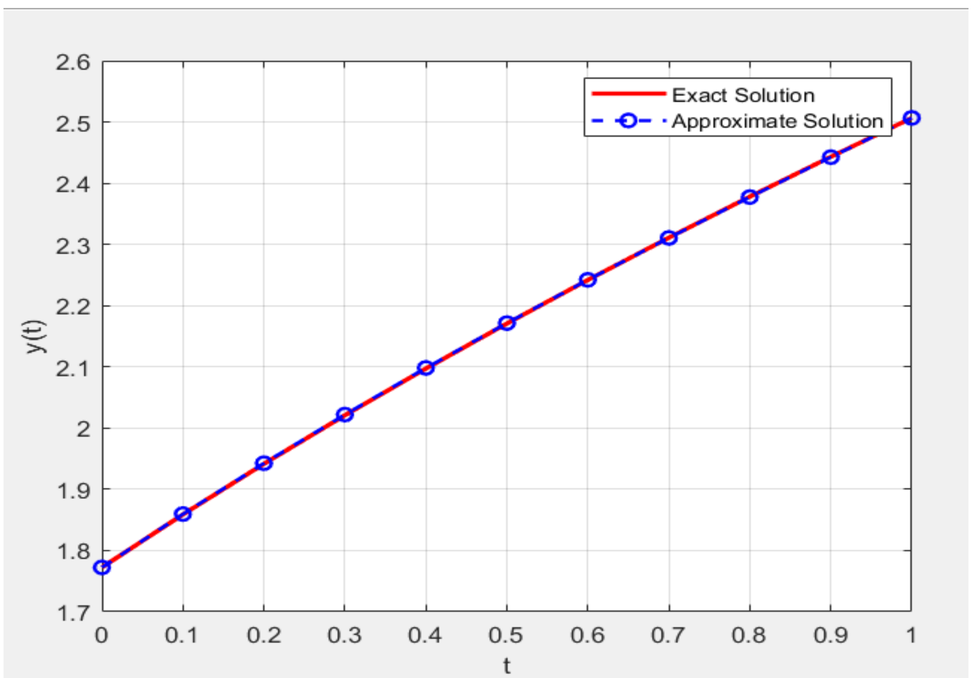


Figure 4.1: comparison Between Exact and the Approximate Solutions Using the Fractional Legendre Wavelet Operational Matrix for $M = 3$

When $M = 4$ and $k = 0$, we get

Now, when $i = 3, j = 4, k = 2$ and $\alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{3,4,2} = \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+2+0}(3+2)!\,(4+0)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= \frac{-80\sqrt{7}\sqrt{9}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+2+1}(3+2)!\,(4+1)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= \frac{960\sqrt{7}\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+2+2}(3+2)!\,(4+2)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-2)!\,(2!)^2\left(2+2-\frac{3}{2}+1\right)}$$

$$= \frac{-21600\sqrt{7}\sqrt{9}}{7\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+2+3}(3+2)!\,(4+3)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-3)!\,(3!)^2\left(2+3-\frac{3}{2}+1\right)}$$

$$= \frac{11200\sqrt{7}\sqrt{9}}{3\sqrt{\pi}}$$

Fifth when $l = 4$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+2+4}(3+2)!\,(4+4)!}{(3-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-4)!\,(4!)^2\left(2+4-\frac{3}{2}+1\right)}$$

$$= \frac{-16800\sqrt{7}\sqrt{9}}{11\sqrt{\pi}}$$

From $l = 0$, $l = 1, l = 2, l = 3$ and $l = 4$, we obtain,

$$\frac{-80\sqrt{7}\sqrt{9}}{\sqrt{\pi}}+\frac{960\sqrt{7}\sqrt{9}}{\sqrt{\pi}}-\frac{21600\sqrt{7}\sqrt{9}}{7\sqrt{\pi}}+\frac{11200\sqrt{7}\sqrt{9}}{3\sqrt{\pi}}-\frac{16800\sqrt{7}\sqrt{9}}{11\sqrt{\pi}}=\frac{80\sqrt{7}\sqrt{9}}{231\sqrt{\pi}}$$

Now, when $i=4, j=0, k=2, l=0$ and $\alpha=3/2$, we get

$$\theta_{4,0,2}=\sqrt{2\times 4+1}\sqrt{2\times 0+1}\frac{(-1)^{4+0+2+0}(4+2)!\,(0+0)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(0-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$=\frac{240\sqrt{9}}{\sqrt{\pi}}$$

Now, when $i=4, j=1, k=2$ and $\alpha=3/2$, we get

First when $l=0$, we find

$$\theta_{4,1,2}=\sqrt{2\times 4+1}\sqrt{2\times 1+1}\frac{(-1)^{4+1+2+0}(4+2)!\,(1+0)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(1-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$=\frac{-240\sqrt{3}\sqrt{9}}{\sqrt{\pi}}$$

Second when $l=1$, we find

$$\theta_{4,1,2}=\sqrt{2\times 4+1}\sqrt{2\times 1+1}\frac{(-1)^{4+1+2+1}(4+2)!\,(1+1)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(1-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$=\frac{288\sqrt{3}\sqrt{9}}{\sqrt{\pi}}$$

From $l=0$ and $l=1$, we obtain,

$$\frac{-240\sqrt{3}\sqrt{9}}{\sqrt{\pi}}+\frac{288\sqrt{3}\sqrt{9}}{\sqrt{\pi}}=\frac{48\sqrt{3}\sqrt{9}}{\sqrt{\pi}}$$

Now, when $i=4, j=2, k=2$ and $\alpha=3/2$, we get

First when $l=0$, we find

$$\theta_{4,2,2}=\sqrt{2\times 4+1}\sqrt{2\times 2+1}\frac{(-1)^{4+2+2+0}(4+2)!\,(2+0)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$=\frac{240\sqrt{5}\sqrt{9}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{4+2+2+1}(4+2)!\,(2+1)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= \frac{-864\sqrt{5}\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{4+2+2+2}(4+2)!\,(2+2)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(2-2)!\,(2!)^2\left(2+2-\frac{3}{2}+1\right)}$$

$$= \frac{4320\sqrt{5}\sqrt{9}}{7\sqrt{\pi}}$$

From $l = 0, l = 1$ and $l = 2$, we obtain,

$$\frac{240\sqrt{5}\sqrt{9}}{\sqrt{\pi}} - \frac{864\sqrt{5}\sqrt{9}}{\sqrt{\pi}} + \frac{4320\sqrt{5}\sqrt{9}}{7\sqrt{\pi}} = \frac{-48\sqrt{5}\sqrt{9}}{7\sqrt{\pi}}$$

Now, when $i = 4, j = 3, k = 2\ and\ \alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{4,3,2} = \sqrt{2 \times 4 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{4+3+2+0}(4+2)!\,(3+0)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(3-0)!\,(0!)^2\left(2+0-\frac{3}{2}+1\right)}$$

$$= \frac{-240\sqrt{7}\sqrt{9}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{4+3+2+1}(4+2)!\,(3+1)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(3-1)!\,(1!)^2\left(2+1-\frac{3}{2}+1\right)}$$

$$= \frac{1728\sqrt{7}\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{4+3+2+2}(4+2)!\,(3+2)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(3-2)!\,(2!)^2\left(2+2-\frac{3}{2}+1\right)}$$

$$= \frac{-21600\sqrt{7}\sqrt{9}}{7\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{4+3+2+3}(4+2)!\,(3+3)!}{(4-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(3-3)!\,(3!)^2\left(2 + 3 - \frac{3}{2} + 1\right)}$$

$$= \frac{1600\sqrt{7}\sqrt{9}}{\sqrt{\pi}}$$

From $l = 0, l = 1, l = 2$ and $l = 3$, we obtain,

$$= \frac{-240\sqrt{7}\sqrt{9}}{\sqrt{\pi}} + \frac{1728\sqrt{7}\sqrt{9}}{\sqrt{\pi}} - \frac{21600\sqrt{7}\sqrt{9}}{7\sqrt{\pi}} + \frac{1600\sqrt{7}\sqrt{9}}{\sqrt{\pi}} = \frac{16\sqrt{7}\sqrt{9}}{7\sqrt{\pi}}$$

Now, when $i = 4, j = 4, k = 2$ and $\alpha = 3/2$, we get

First when $l = 0$, we find

$$\theta_{4,4,2} = \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+2+0}(4+2)!\,(4+0)!}{(4-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(4-0)!\,(0!)^2\left(2 + 0 - \frac{3}{2} + 1\right)}$$

$$= \frac{2160}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+2+1}(4+2)!\,(4+1)!}{(4-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(4-1)!\,(1!)^2\left(2 + 1 - \frac{3}{2} + 1\right)}$$

$$= \frac{-25920}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+2+2}(4+2)!\,(4+2)!}{(4-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(4-2)!\,(2!)^2\left(2 + 2 - \frac{3}{2} + 1\right)}$$

$$= \frac{583200}{7\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+2+3}(4+2)!\,(4+3)!}{(4-2)!\,2!\,\Gamma\left(2 - \frac{3}{2} + 1\right)(4-3)!\,(3!)^2\left(2 + 3 - \frac{3}{2} + 1\right)}$$

$$= \frac{-100800}{\sqrt{\pi}}$$

Fifth when $l = 4$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+2+4}(4+2)!\,(4+4)!}{(4-2)!\,2!\,\Gamma\left(2-\frac{3}{2}+1\right)(4-4)!\,(4!)^2\left(2+4-\frac{3}{2}+1\right)}$$

$$= \frac{453600}{11\sqrt{\pi}}$$

From $l = 0, l = 1, l = 2, l = 3$ and $l = 4$, we obtain,

$$\frac{2160}{\sqrt{\pi}} - \frac{25920}{\sqrt{\pi}} + \frac{583200}{7\sqrt{\pi}} - \frac{100800}{\sqrt{\pi}} + \frac{453600}{11\sqrt{\pi}} = \frac{-720}{77\sqrt{\pi}}$$

$$\boldsymbol{D^{3/2}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\mathbf{16\sqrt{5}}}{\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{16\sqrt{5}\sqrt{3}}}{\mathbf{5}\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{-16}}{\mathbf{7}\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{16\sqrt{5}\sqrt{7}}}{\mathbf{105}\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{26\sqrt{5}\sqrt{9}}}{\mathbf{231}\boldsymbol{\sqrt{\pi}}} \\ \frac{\mathbf{-80\sqrt{7}}}{\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{-16\sqrt{7}\sqrt{3}}}{\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{16\sqrt{7}\sqrt{5}}}{\mathbf{7}\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{-16}}{\mathbf{3}\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{80\sqrt{9}\sqrt{7}}}{\mathbf{231}\boldsymbol{\sqrt{\pi}}} \\ \frac{\mathbf{240\sqrt{9}}}{\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{48\sqrt{9}\sqrt{3}}}{\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{-48\sqrt{9}\sqrt{5}}}{\mathbf{7}\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{16\sqrt{9}\sqrt{7}}}{\mathbf{7}\boldsymbol{\sqrt{\pi}}} & \frac{\mathbf{-720}}{\mathbf{77}\boldsymbol{\sqrt{\pi}}} \end{bmatrix}$$

Now, when $i = 3, j = 4, k = 3$ and $\alpha = 5/2$, we get,

First when $l = 0$, we find

$$\theta_{3,4,3} = \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+3+0}(3+3)!\,(4+0)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \frac{160\sqrt{7\sqrt{9}}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+3+1}(3+3)!\,(4+1)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{-1920\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 3 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{3+4+3+2}(3+3)!\,(4+2)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-2)!\,(2!)^2\left(3+2-\frac{5}{2}+1\right)}$$

$$= \frac{43200\sqrt{7}\sqrt{9}}{7\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2\times 3+1}\sqrt{2\times 4+1}\frac{(-1)^{3+4+3+3}(3+3)!\,(4+3)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-3)!\,(3!)^2\left(3+3-\frac{5}{2}+1\right)}$$

$$= \frac{-22400\sqrt{7}\sqrt{9}}{3\sqrt{\pi}}$$

Fifth when $l = 4$, we find

$$= \sqrt{2\times 3+1}\sqrt{2\times 4+1}\frac{(-1)^{3+4+3+4}(3+3)!\,(4+4)!}{(3-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-4)!\,(4!)^2\left(3+4-\frac{5}{2}+1\right)}$$

$$= \frac{33600\sqrt{7}\sqrt{9}}{11\sqrt{\pi}}$$

From $l = 0, l = 1, l = 2, l = 3$ and $l = 4$, we obtain,

$$\frac{160\sqrt{7\sqrt{9}}}{\sqrt{\pi}} - \frac{1920\sqrt{7}\sqrt{9}}{\sqrt{\pi}} + \frac{43200\sqrt{7}\sqrt{9}}{7\sqrt{\pi}} - \frac{22400\sqrt{7}\sqrt{9}}{3\sqrt{\pi}} + \frac{33600\sqrt{7}\sqrt{9}}{11\sqrt{\pi}}$$

$$= \frac{-160\sqrt{7}\sqrt{9}}{\sqrt{\pi}}$$

Now, when $i = 4, j = 0, k = 3, l = 0 \; and \; \alpha = 5/2$, we get

$$\theta_{4,0,3} = \sqrt{2\times 4+1}\sqrt{2\times 0+1}\frac{(-1)^{4+0+3+0}(4+3)!\,(0+0)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(0-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \frac{-3360}{\sqrt{\pi}}$$

Now, when $i = 4, j = 1, k = 3 \; and \; \alpha = 5/2$, we get

First when $l = 0$, we find

$$\theta_{4,1,3} = \sqrt{2\times 4+1}\sqrt{2\times 1+1}\frac{(-1)^{4+1+3+0}(4+3)!\,(1+0)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(1-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \frac{1120\sqrt{3}\sqrt{9}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2\times 4+1}\sqrt{2\times 1+1}\frac{(-1)^{4+1+3+1}(4+3)!\,(1+1)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(1-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{-1344\sqrt{3}\,\sqrt{9}}{\sqrt{\pi}}$$

From $l = 0$and $l = 1$, we obtain,

$$\frac{1120\sqrt{3}\,\sqrt{9}}{\sqrt{\pi}} - \frac{1344\sqrt{3}\,\sqrt{9}}{\sqrt{\pi}} = \frac{-224\sqrt{3}\,\sqrt{9}}{\sqrt{\pi}}$$

Now, when $i = 4, j = 2, k = 3\ and\ \alpha = 5/2$, we get

First when $l = 0$, we find

$$\theta_{4,2,3} = \sqrt{2\times 4+1}\sqrt{2\times 2+1}\frac{(-1)^{4+2+3+0}(4+3)!\,(2+0)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(2-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \frac{-1120\sqrt{5}\,\sqrt{9}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2\times 4+1}\sqrt{2\times 2+1}\frac{(-1)^{4+2+3+1}(4+3)!\,(2+1)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(2-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{4032\sqrt{5}\,\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2\times 4+1}\sqrt{2\times 2+1}\frac{(-1)^{4+2+3+2}(4+3)!\,(2+2)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(2-2)!\,(2!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{-2880\sqrt{5}\,\sqrt{9}}{\sqrt{\pi}}$$

From $l = 0, l = 1$and $l = 2$, we obtain,

$$\frac{-1120\sqrt{5}\,\sqrt{9}}{\sqrt{\pi}} + \frac{4032\sqrt{5}\,\sqrt{9}}{\sqrt{\pi}} - \frac{2880\sqrt{5}\,\sqrt{9}}{\sqrt{\pi}} = \frac{32\sqrt{5}\,\sqrt{9}}{\sqrt{\pi}}$$

Now, when $i = 4, j = 3, k = 3\ and\ \alpha = 5/2$, we get:

First when $l = 0$, we find

$$\theta_{4,3,3} = \sqrt{2\times 4+1}\sqrt{2\times 3+1}\frac{(-1)^{4+3+3+0}(4+3)!\,(3+0)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \frac{1120\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2\times 4+1}\sqrt{2\times 3+1}\frac{(-1)^{4+3+3+1}(4+3)!\,(3+1)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{-8064\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2\times 4+1}\sqrt{2\times 3+1}\frac{(-1)^{4+3+3+2}(4+3)!\,(2+2)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-2)!\,(2!)^2\left(3+2-\frac{5}{2}+1\right)}$$

$$= \frac{14400\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2\times 4+1}\sqrt{2\times 3+1}\frac{(-1)^{4+3+3+3}(4+3)!\,(3+3)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-3)!\,(3!)^2\left(3+3-\frac{5}{2}+1\right)}$$

$$= \frac{-22400\sqrt{7}\,\sqrt{9}}{3\sqrt{\pi}}$$

$$= \sqrt{2\times 4+1}\sqrt{2\times 3+1}\frac{(-1)^{4+3+3+1}(4+3)!\,(3+1)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{-8064\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2\times 4+1}\sqrt{2\times 3+1}\frac{(-1)^{4+3+3+2}(4+3)!\,(2+2)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-2)!\,(2!)^2\left(3+2-\frac{5}{2}+1\right)}$$

$$= \frac{14400\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 3 + 1}\frac{(-1)^{4+3+3+3}(4+3)!\,(3+3)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(3-3)!\,(3!)^2\left(3+3-\frac{5}{2}+1\right)}$$

$$= \frac{-22400\sqrt{7}\,\sqrt{9}}{3\sqrt{\pi}}$$

From $l = 0, l = 1, l = 2$and $l = 3$, we obtain:

$$\frac{1120\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}} - \frac{8064\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}} + \frac{14400\sqrt{7}\,\sqrt{9}}{\sqrt{\pi}} - \frac{22400\sqrt{7}\,\sqrt{9}}{3\sqrt{\pi}} = \frac{-32\sqrt{7}\,\sqrt{9}}{3\sqrt{\pi}}$$

Now, when $i = 4, j = 4, k = 3\ and\ \alpha = 5/2$, we get

First when $l = 0$, we find

$$\theta_{4,4,3} = \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+3+0}(4+3)!\,(4+0)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-0)!\,(0!)^2\left(3+0-\frac{5}{2}+1\right)}$$

$$= \frac{-10080}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+3+1}(4+3)!\,(4+1)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-1)!\,(1!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{120960}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+3+2}(4+3)!\,(4+2)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-2)!\,(2!)^2\left(3+1-\frac{5}{2}+1\right)}$$

$$= \frac{-388800}{\sqrt{\pi}}$$

Fourth when $l = 3$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+3+3}(4+3)!\,(4+3)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-3)!\,(3!)^2\left(3+3-\frac{5}{2}+1\right)}$$

$$= \frac{1411200}{3\sqrt{\pi}}$$

The fifth when $l = 4$, we find

$$= \sqrt{2 \times 4 + 1}\sqrt{2 \times 4 + 1}\frac{(-1)^{4+4+3+4}(4+3)!\,(4+4)!}{(4-3)!\,3!\,\Gamma\left(3-\frac{5}{2}+1\right)(4-4)!\,(4!)^2\left(3+4-\frac{5}{2}+1\right)}$$

$$= \frac{-2116800}{11\sqrt{\pi}}$$

From $l = 0, l = 1, l = 2, l = 3$ and $l = 4$, we obtain:

$$\frac{-10080}{\sqrt{\pi}} + \frac{120960}{\sqrt{\pi}} - \frac{388800}{\sqrt{\pi}} + \frac{1411200}{3\sqrt{\pi}} - \frac{2116800}{11\sqrt{\pi}} = \frac{480}{11\sqrt{\pi}}$$

$$\boldsymbol{D}^{5/2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\mathbf{160\sqrt{7}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{32\sqrt{7}\sqrt{3}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{-32\sqrt{7}\sqrt{5}}}{\mathbf{7}\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{32}}{\mathbf{3}\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{-160\sqrt{7}\sqrt{9}}}{\sqrt{\boldsymbol{\pi}}} \\ \frac{\mathbf{-3360}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{-224\sqrt{3}\sqrt{9}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{32\sqrt{5}\sqrt{9}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{-32\sqrt{7}\sqrt{9}}}{\sqrt{\boldsymbol{\pi}}} & \frac{\mathbf{480}}{\mathbf{11}\sqrt{\boldsymbol{\pi}}} \end{bmatrix}$$

Table 4.2: Comparison Between Exact Solution $y(t)$ and the Approximate Solution $y_{LWOMM}(t)$ for $M = 4$

t	Exact Solution	Approximate Solution	Absolute error
0.0	1.77245385	1.77245017	3.6799e-06
0.1	1.85896528	1.85865759	0.0003076896
0.2	1.94162591	1.94212836	0.0005024513
0.3	2.02090832	2.02282296	0.0019146374
0.4	2.09719568	2.10070186	0.0035061777
0.5	2.17080376	2.17572553	0.0049217632
0.6	2.24199649	2.24785445	0.0058579596
0.7	2.31099708	2.31704909	0.0060520074
0.8	2.37799638	2.38326993	0.0052735513
0.9	2.44315903	2.44647744	0.0033184145
1.0	2.50662827	2.50663210	3.8303e-06

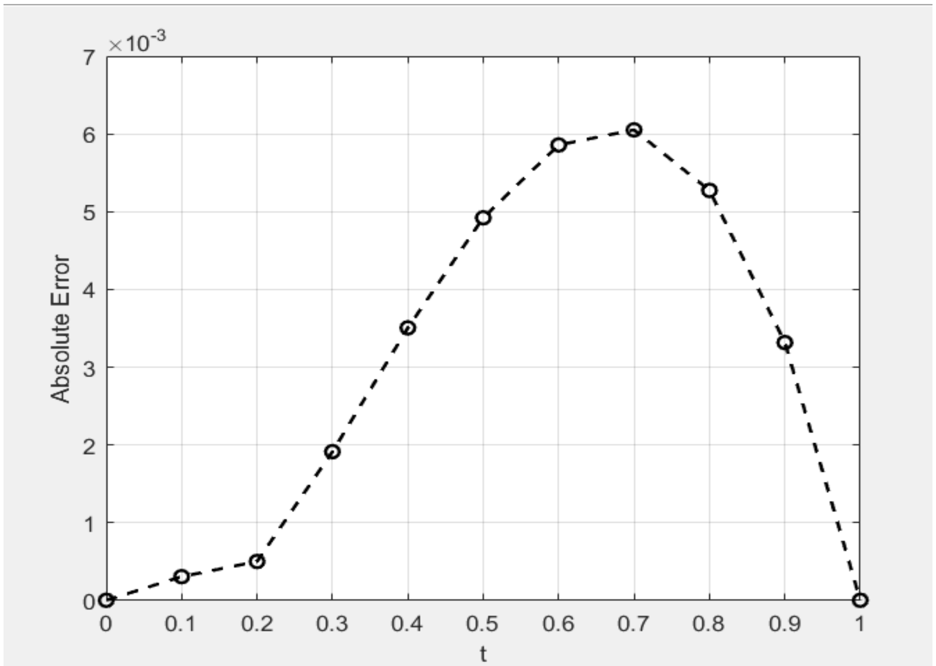


Figure 4.2: Absolute Error between Exact and Approximate Solutions for Equation (4.24) at $M = 4$

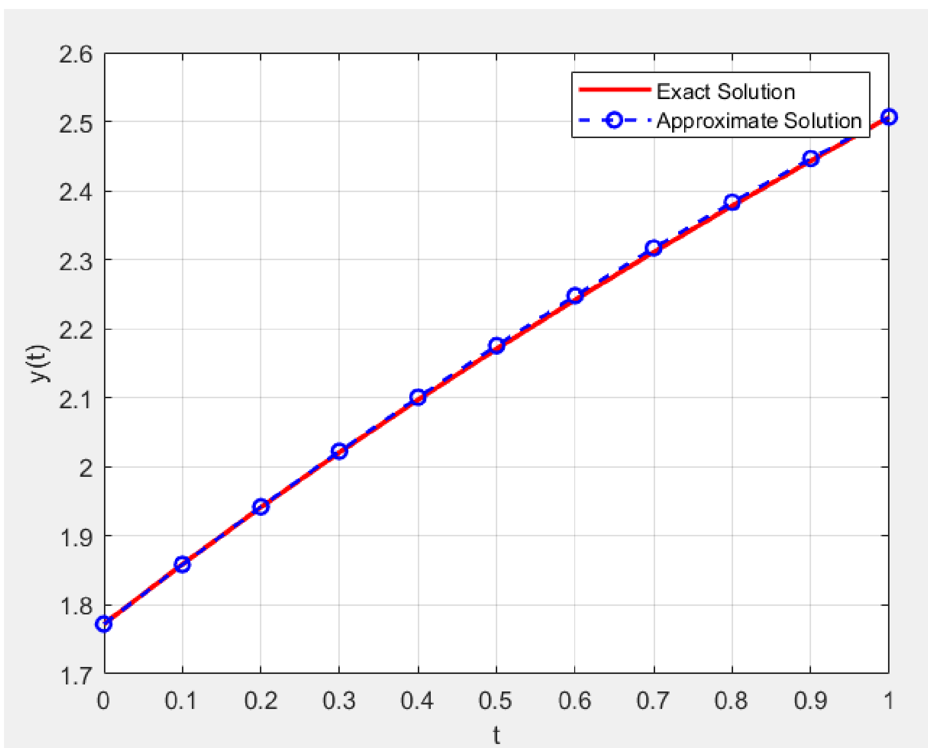


Figure 4.2: Comparison Between the Exact and the Approximate Solutions Using the Fractional Legendre Wavelet Operational Matrix for $M = 4$.

This example 1 **(Secer et al., 2019)** was re-solved using the proposed Legendre Wavelet Operational Matrix (LWOM) method for both $M = 3$ and $M = 4$. The results indicate that increasing the value of M improves the accuracy, with $M = 4$ yielding significantly better results than $M = 3$. It was also observed that the Fourth row of the $D^{3/2}$ matrix in the reference **(Secer et al., 2019**) contains a computational error, which negatively affected their solution.

Here, the matrix was manually constructed correctly in this work, leading to highly accurate numerical results, as show in Figures (4.1) and Figures (4.2).

Furthermore, the comparison plot between the exact and approximate solutions shows a strong agreement, with nearly identical curves, confirming the effectiveness of the method.

Table (4.1) and Table (4.2) present the approximate solutions, exact values, and corresponding absolute errors for both values of M.

Example 2: (Saadatmandi, 2014)

consider the following linear initial values problem that consists the fractional differential equations:

$$D^2 y(t) + D^{\frac{3}{2}} y(t) + y(t) = 1 + t \tag{4.25}$$

And the initial conditions,

$$y(0) = 0, \frac{dy}{dt}(0) = 1$$

The exact solution of this problem is given as,

$$y(t) = 1 + t$$

Solution:

We set $M = 2\ and\ k = 0$; two fractional operational matrices $D^{3/2} and\ D$, whose results were presented in the example 1, are used in the subsequent computations.

$$\boldsymbol{D} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{2\sqrt{3}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2\sqrt{15}} & \mathbf{0} \end{bmatrix}$$

Next, the fractional operational matrices D^2will be analyzed and constructed as

$D \times D = D^2$, it can be written in the form of a theorem (4.1) as,

$$D^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 \end{bmatrix}$$

Table 4.3: Comparison Between the of Exact Solution $y(t)$ and the Approximate Solution $y_{LWOMM}(t)$ for $M = 2$

t	Exact Solution	Approximate Solution	Absolute error
0.0	1.000000000	1.000000000	4.4e-16
0.1	1.100000000	1.100000000	5.8e-14
0.2	1.200000000	1.200000000	2.3e-13
0.3	1.300000000	1.300000000	5.3e-13
0.4	1.400000000	1.400000000	9.4e-13
0.5	1.500000000	1.500000000	1.5e-12
0.6	1.600000000	1.600000000	2.1e-12
0.7	1.700000000	1.700000000	2.9e-12
0.8	1.800000000	1.800000000	3.8e-12
0.9	1.900000000	1.900000000	4.8e-12
1.0	2.000000000	2.000000000	5.9e-12

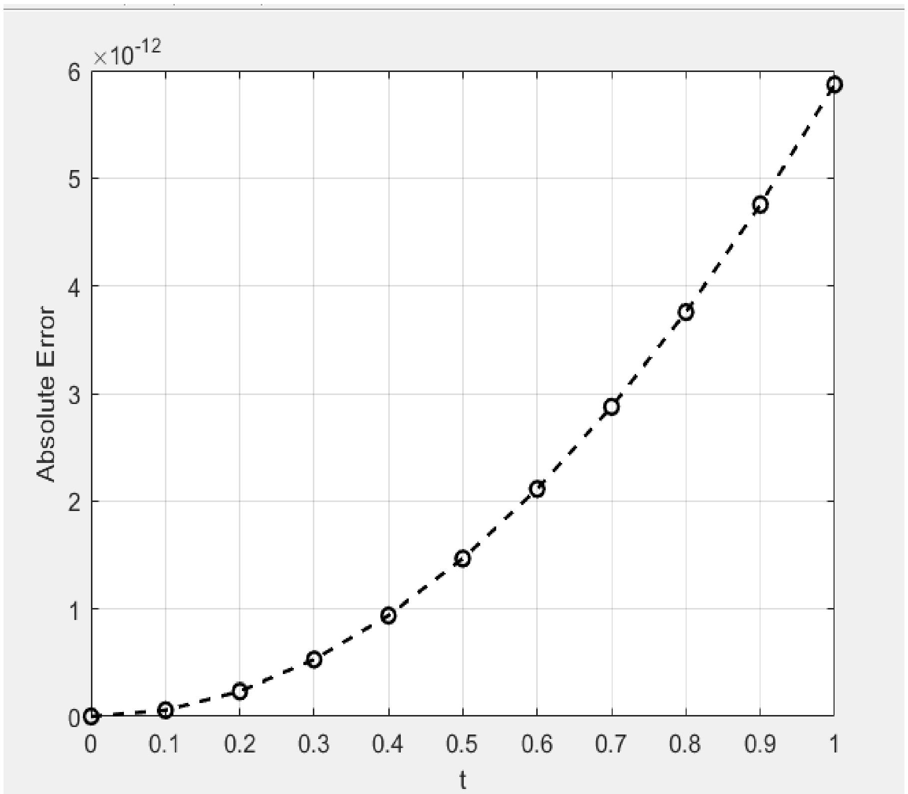


Figure 4.3: Absolute Error between Exact and Approximate Solutions for Equation (4.25) at $M = 2$

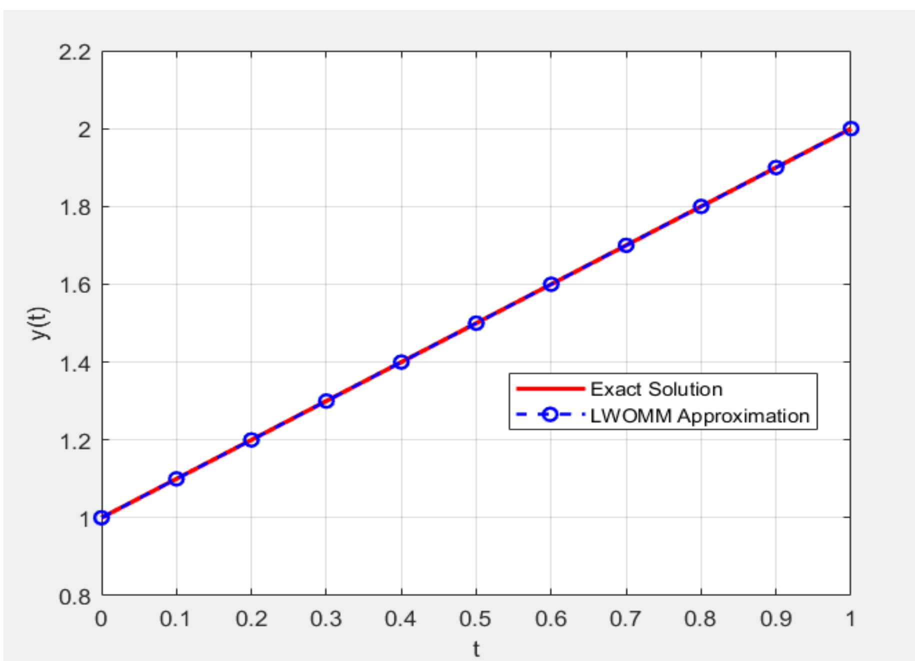


Figure 4.3: comparison Between the Exact and the Approximate Solutions Using the Fractional Legendre Wavelet Operational Matrix for $M = 2$

Now, we take $M = 3$ and $k = 0$ then the value D^2 is

$$\boldsymbol{D^2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{4\sqrt{3}\sqrt{15}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{4\sqrt{15}\sqrt{35}} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Using the same code but changing $M = 3$ and $k = 0$ we get that

Table 4.4: Comparison Between the Exact Solution $y(t)$ and the Approximate Solution $y_{LWOMM}(t)$ for $M = 3$

$\boldsymbol{t}$	**Exact Solution**	**Approximate Solution**	**Absolute error**
0.0	1.000000000	1.000000000	3.3e-16
0.1	1.100000000	1.100000000	4.4e-13
0.2	1.200000000	1.200000000	8.3e-13
0.3	1.300000000	1.300000000	1.2e-12
0.4	1.400000000	1.400000000	1.5e-12
0.5	1.500000000	1.500000000	1.7e-12
0.6	1.600000000	1.600000000	1.9e-12
0.7	1.700000000	1.700000000	2.0e-12
0.8	1.800000000	1.800000000	2.1e-12
0.9	1.900000000	1.900000000	2.1e-12
1.0	2.000000000	2.000000000	2.0e-12

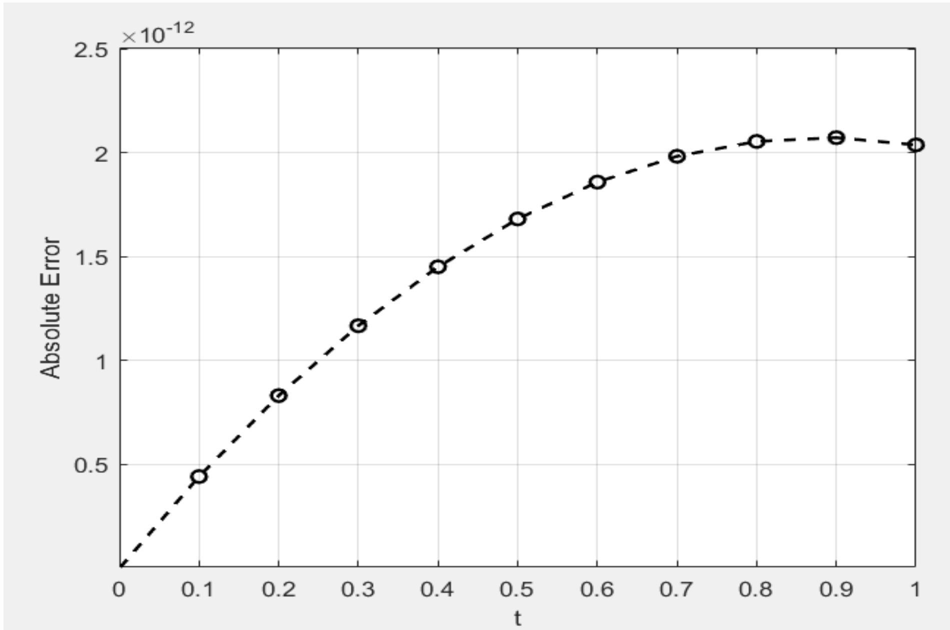


Figure 4.4: Absolute Error Between Exact and Approximate Solutions for Equation (4.25) at $M = 3$.

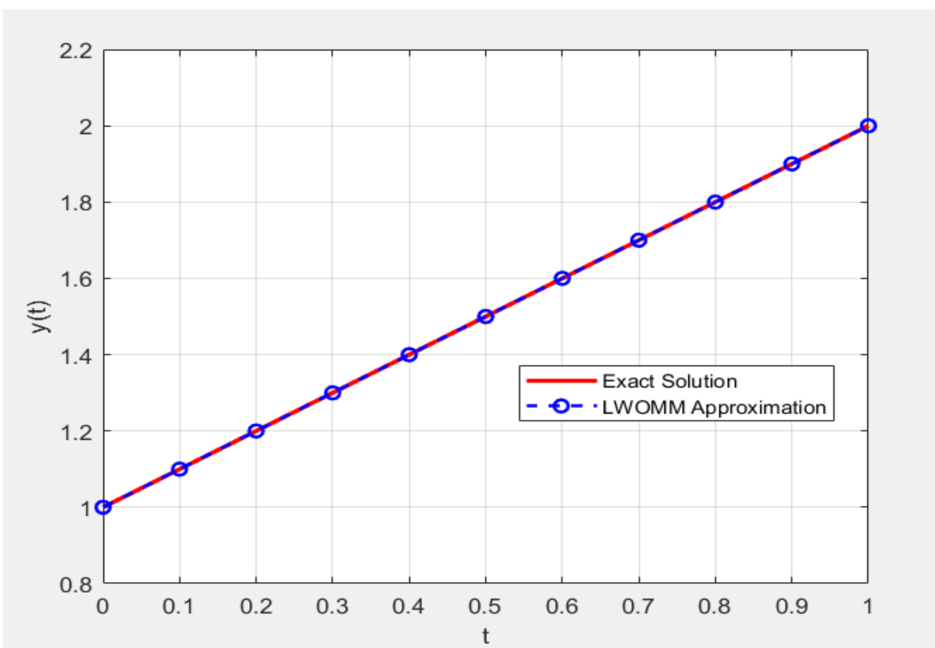


Figure 4.4: comparison Between the Exact and the Approximate Solutions Using the Fractional Legendre Wavelet Operational Matrix for $M = 3$.

This example2 **(Saadatmandi, 2014)** was re-solved using the Legendre Wavelet Operational Matrix (LWOM) for two different values of M, namely $M = 2$ and $M = 3$, to examine the effect of increasing on the solution accuracy.

The numerical results were accurate in both cases, with better accuracy achieved at $M = 3$, as confirmed by the absolute error values.

As shown in Figure (4.3) and Figure (4.4), the comparison between the exact and approximate solutions reveals a close match, highlighting the efficiency of the proposed method.

Table (4.3) and Table (4.4) present the approximate solutions, exact values, and corresponding absolute errors for both values of M.

Example 3:

consider the following linear initial value problem consists of the fractional differential equations:

$$4(t+1)D^3y(t) + 4D^{\frac{1}{2}}y(t) + \frac{1}{\sqrt{t+1}}y(t) = \sqrt{t} + \sqrt{\pi} \quad (4.26)$$

And the initial condition,

$$y(0) = \sqrt{\pi}, \frac{dy}{dt}(0) = \frac{\sqrt{\pi}}{2}, y(1) = \sqrt{2\pi}$$

The exact solution of this problem is

$$y(t) = \sqrt{\pi(t+1)}$$

Solution:

We set $M = 3\ and\ k = 0$; and the matrix D^3 will be derived using the same approach as for D^2 Next, the fractional operational matrices D^3will be analyzed and constructed as $D \times D^2 = D^3$, it can be written in the form of a theorem (4.1) as,

$$\boldsymbol{D^3} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{8\sqrt{3}\sqrt{15}\sqrt{35}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

And now we are going to find matrix $D^{\frac{1}{2}}$ when $i = 1, j = 0, k = 1, l = 0\ and\ \alpha = 0.5$, we get

$$\theta_{1,0,1} = \sqrt{2\times 1+1}\sqrt{2\times 0+1}\frac{(-1)^{1+0+1+0}(1+1)!\,(0+0)!}{(1-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(0-0)!\,(0!)^2\left(1+0-\frac{1}{2}+1\right)}$$

$$= \frac{8\sqrt{3}}{3\sqrt{\pi}}$$

Now, when $i = 1, j = 1, k = 1, l = 0 \; and \; \alpha = 0.5$, we get

First when $l = 0$, we find

$\theta_{1,1,1}$

$$= \sqrt{2 \times 1 + 1}\sqrt{2 \times 1 + 1}\frac{(-1)^{1+1+1+0}(1+1)!\,(1+0)!}{(1-1)!\,1!\,\Gamma\left(1 - \frac{1}{2} + 1\right)(1-0)!\,(0!)^2\left(1 + 0 - \frac{1}{2} + 1\right)}$$

$$= \frac{-8}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 1 + 1}\sqrt{2 \times 1 + 1}\frac{(-1)^{1+1+1+1}(1+1)!\,(1+1)!}{(1-1)!\,1!\,\Gamma\left(1 - \frac{1}{2} + 1\right)(1-1)!\,(1!)^2\left(1 + 1 - \frac{1}{2} + 1\right)}$$

$$= \frac{48}{5\sqrt{\pi}}$$

From $l = 0$and $l = 1$, we obtain,

$$\frac{-8}{\sqrt{\pi}} + \frac{48}{5\sqrt{\pi}} = \frac{8}{5\sqrt{\pi}}$$

Now, when $i = 1, j = 2, k = 1 \; and \; \alpha = 0.5$, we get

First when $l = 0$, we find

$$\theta_{1.2,1} = \sqrt{2 \times 1 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{1+2+1+0}(1+1)!\,(2+0)!}{(1-1)!\,1!\,\Gamma\left(1 - \frac{1}{2} + 1\right)(2-0)!\,(0!)^2\left(1 + 0 - \frac{1}{2} + 1\right)}$$

$$= \frac{8\sqrt{3}\sqrt{5}}{3\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2 \times 1 + 1}\sqrt{2 \times 2 + 1}\frac{(-1)^{1+2+1+1}(1+1)!\,(2+1)!}{(1-1)!\,1!\,\Gamma\left(1 - \frac{1}{2} + 1\right)(2-1)!\,(1!)^2\left(1 + 1 - \frac{1}{2} + 1\right)}$$

$$= \frac{-48\sqrt{3}\sqrt{5}}{5\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2\times 1+1}\sqrt{2\times 2+1}\frac{(-1)^{1+2+1+2}(1+1)!\,(2+2)!}{(1-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(2-2)!\,(2!)^2\left(1+2-\frac{1}{2}+1\right)}$$

$$= \frac{48\sqrt{3}\sqrt{5}}{7\sqrt{\pi}}$$

From $l = 0, l = 1$and $l = 2$, we obtain:

$$\frac{8\sqrt{3}\sqrt{5}}{3\sqrt{\pi}} - \frac{48\sqrt{3}\sqrt{5}}{5\sqrt{\pi}} + \frac{48\sqrt{3}\sqrt{5}}{7\sqrt{\pi}} = \frac{-8}{105\sqrt{\pi}}$$

And with the same steps we find that:

Now, when $i = 1, j = 3, k = 1 and\ \alpha = 0.5, when\ l = 0, l = 1, l = 2\ and\ l = 3$, we get

$$\theta_{1,3,1} == \frac{8\sqrt{3}\sqrt{7}}{315\sqrt{\pi}}$$

Now, when $i = 2, j = 0, k = 1, l = 0\ and\ \alpha = 0.5$, we get

$$\theta_{2,0,1} = \sqrt{2\times 2+1}\sqrt{2\times 0+1}\frac{(-1)^{2+0+1+0}(2+1)!\,(0+0)!}{(2-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(0-0)!\,(0!)^2\left(1+0-\frac{1}{2}+1\right)}$$

$$= \frac{-8\sqrt{5}}{\sqrt{\pi}}$$

Now, when $i = 2, j = 1, k = 1 and\ \alpha = 0.5$, we get

First when $l = 0$, we find

$$\theta_{2,1,1} = \sqrt{2\times 2+1}\sqrt{2\times 1+1}\frac{(-1)^{2+1+1+0}(2+1)!\,(1+0)!}{(2-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(1-0)!\,(0!)^2\left(1+0-\frac{1}{2}+1\right)}$$

$$= \frac{8\sqrt{5}\,\sqrt{3}}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2\times 2+1}\sqrt{2\times 1+1}\frac{(-1)^{2+1+1+1}(2+1)!\,(1+1)!}{(2-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(1-1)!\,(1!)^2\left(1+1-\frac{1}{2}+1\right)}$$

$$= \frac{-48\sqrt{5}\,\sqrt{3}}{5\sqrt{\pi}}$$

From $l = 0$and $l = 1$, we obtain,

$$\frac{8\sqrt{5}\,\sqrt{3}}{\sqrt{\pi}} - \frac{48\sqrt{5}\,\sqrt{3}}{5\sqrt{\pi}} = \frac{-8\sqrt{5}\,\sqrt{3}}{5\sqrt{\pi}}$$

Now, when $i = 2, j = 2, k = 1 and\ \alpha = 0.5$, we get

First when $l = 0$, we find

$$\theta_{2,2,1} = \sqrt{2\times 2+1}\sqrt{2\times 2+1}\frac{(-1)^{2+2+1+0}(2+1)!\,(2+0)!}{(2-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(2-0)!\,(0!)^2\left(1+0-\frac{1}{2}+1\right)}$$

$$= \frac{-40}{\sqrt{\pi}}$$

Second when $l = 1$, we find

$$= \sqrt{2\times 2+1}\sqrt{2\times 2+1}\frac{(-1)^{2+2+1+1}(2+1)!\,(2+1)!}{(2-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(2-1)!\,(1!)^2\left(1+1-\frac{1}{2}+1\right)}$$

$$= \frac{144}{\sqrt{\pi}}$$

Third when $l = 2$, we find

$$= \sqrt{2\times 2+1}\sqrt{2\times 2+1}\frac{(-1)^{2+2+1+2}(2+1)!\,(2+2)!}{(2-1)!\,1!\,\Gamma\left(1-\frac{1}{2}+1\right)(2-2)!\,(2!)^2\left(1+2-\frac{1}{2}+1\right)}$$

$$= \frac{-720}{7\sqrt{\pi}}$$

From $l = 0, l = 1$and $l = 2$, we obtain,

$$\frac{-40}{\sqrt{\pi}} + \frac{144}{\sqrt{\pi}} - \frac{-720}{7\sqrt{\pi}} = \frac{8}{7\sqrt{\pi}}$$

And with the same steps we find that:

Now, when $i = 2, j = 3, k = 1 and\ \alpha = 0.5, when\ l = 0, l = 1, l = 2, l = 3$, we get

$$\theta_{2,3,1} = \frac{-8\sqrt{5}\sqrt{7}}{105\sqrt{\pi}}$$

Now, when $i = 3, j = 0, k = 1, l = 0 \; and \; \alpha = 0.5$, we get

$$\theta_{3,0,1} = \frac{-16\sqrt{3}}{\sqrt{\pi}}$$

Now, when $i = 3, j = 1, k = 1, l = 0 \,, l = 1 \; and \; \alpha = 0.5$, we get

$$\theta_{3,1,1} = \frac{16\sqrt{3}\sqrt{7}}{\sqrt{\pi}}$$

Now, when $i = 3, j = 2, k = 1, l = 0 \,, l = 1, l = 2 \; and \; \alpha = 0.5$, we get

$$\theta_{3,2,1} = \frac{-16\sqrt{5}\sqrt{7}}{35\sqrt{\pi}}$$

Now, when $i = 3, j = 3, k = 1, l = 0 \,, l = 1, l = 2, l = 3 \; and \; \alpha = 0.5$, we get

$$\theta_{3,3,1} = \frac{-16}{15\sqrt{\pi}}$$

Thus, we obtain

$$\boldsymbol{D^{1/2}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\frac{8\sqrt{3}}{3\sqrt{\pi}}} & \boldsymbol{\frac{8}{5\sqrt{\pi}}} & \boldsymbol{\frac{-8}{105\sqrt{\pi}}} & \frac{8\sqrt{3}\sqrt{7}}{315\sqrt{\pi}} \\ \boldsymbol{\frac{-8\sqrt{5}}{\sqrt{\pi}}} & \boldsymbol{\frac{-8\sqrt{5}\,\sqrt{3}}{5\sqrt{\pi}}} & \boldsymbol{\frac{8}{7\sqrt{\pi}}} & \frac{-8\sqrt{5}\sqrt{7}}{105\sqrt{\pi}} \\ \frac{-16\sqrt{3}}{\sqrt{\pi}} & \frac{16\sqrt{3}\sqrt{7}}{\sqrt{\pi}} & \frac{-16\sqrt{5}\sqrt{7}}{35\sqrt{\pi}} & \frac{-16}{15\sqrt{\pi}} \end{bmatrix}$$

Table 4.5: Comparison Between Exact Solution $y(t)$ and the Approximate Solution $y_{LWOMM}(t)$ for $M = 3$

t	**Exact Solution**	**Approximate Solution**	**Absolute error**
0.0	1.7724538509	1.7724538987	4.8e-08
0.1	1.8589652818	1.8595560634	5.9e-04
0.2	1.9416259126	1.9445968705	3.0e-03
0.3	2.0209083229	2.0272089353	6.3e-03
0.4	2.0971956788	2.1070248726	9.8e-03
0.5	2.1708037637	2.1836772976	1.3e-02
0.6	2.2419964866	2.2567988253	1.5e-02
0.7	2.3109970816	2.3260220707	1.5e-02

0.8	2.3779963786	2.3909796489	1.3e-02
0.9	2.4431590292	2.4513041749	8.1e-03
1.0	2.5066282746	2.5066282637	1.1e-08

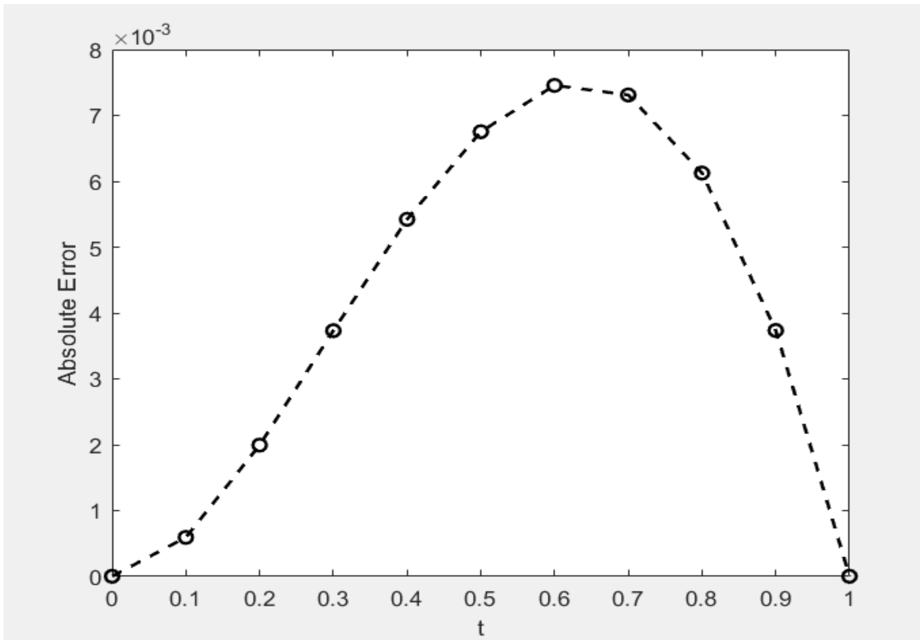


Figure 4.5: Absolute Error Between Exact and Approximate Solutions for equation (4.26) at $M = 3$

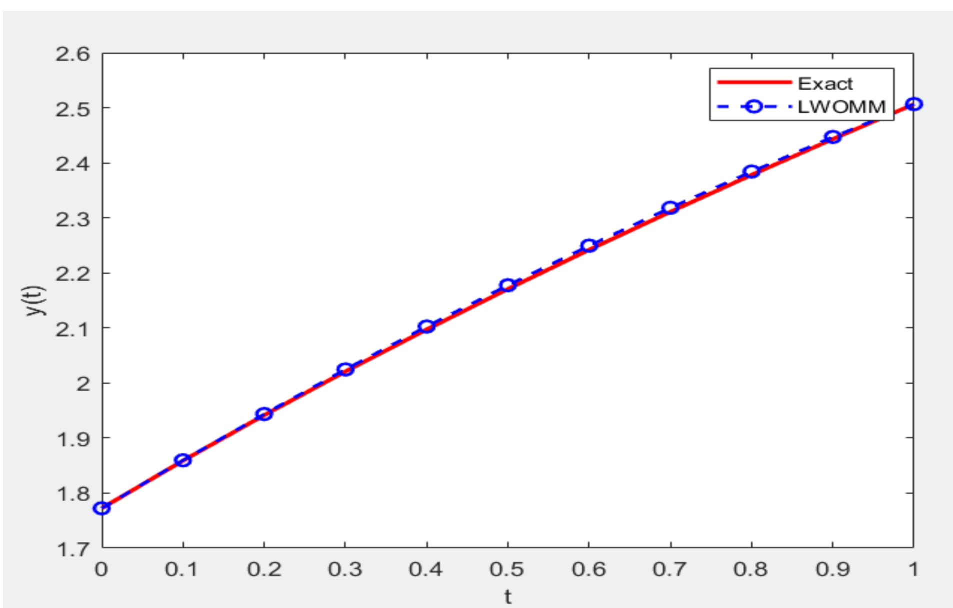


Figure 4.6: Comparison Between the Exact and the Approximate Solutions Using the Fractional Legendre Wavelet Operational Matrix for $M = 3$

This example 3, was solved using the Legendre wavelet operational matrix method **(LWOM)** for $M = 3$. The numerical results and the comparison between the exact and approximate solutions demonstrated excellent agreement as shown in Table (4.5) and Figures (4.5) and (4.6). This confirms the reliability and efficiency of the proposed fractional approach, even with a relatively small value of M. The success of this example adds further evidence to the accuracy and applicability of the method in solving fractional differential equations.

Example 4:

consider the following linear initial values problem that consists the fractional differential equations:

$$\frac{d^2y}{dt^2} + 4(t+1)\, D^{\frac{3}{2}} y(t) + y(t) = 1 + t \tag{4.27}$$

And the initial conditions,

$$y(0) = 0, \frac{dy}{dt}(0) = 1$$

The exact solution of this problem is given as,

$$y(t) = 1 + t$$

Solution:

We set $M = 3\ and\ k = 0$; the matrices D and $D^{\frac{3}{2}}$ have been defined in the previous examples, found the matrix D^2in example 2.

Table 4.6: Comparison Between Exact Solution $y(t)$ and the Approximate Solution $y_{LWOMM}(t)$ for $M = 3$

t	Exact Solution	Approximate Solution	Absolute error
0.0	1.000000000	1.000000000	2.2e-16
0.1	1.100000000	1.100000000	2.5e-13
0.2	1.200000000	1.200000000	4.2e-13
0.3	1.300000000	1.300000000	5.2e-13
0.4	1.400000000	1.400000000	5.5e-13
0.5	1.500000000	1.500000000	5.0e-13
0.6	1.600000000	1.600000000	3.8e-13

0.7	1.700000000	1.700000000	1.9e-13
0.8	1.800000000	1.800000000	7.0e-14
0.9	1.900000000	1.900000000	4.1e-13
1.0	2.000000000	2.000000000	8.1e-13

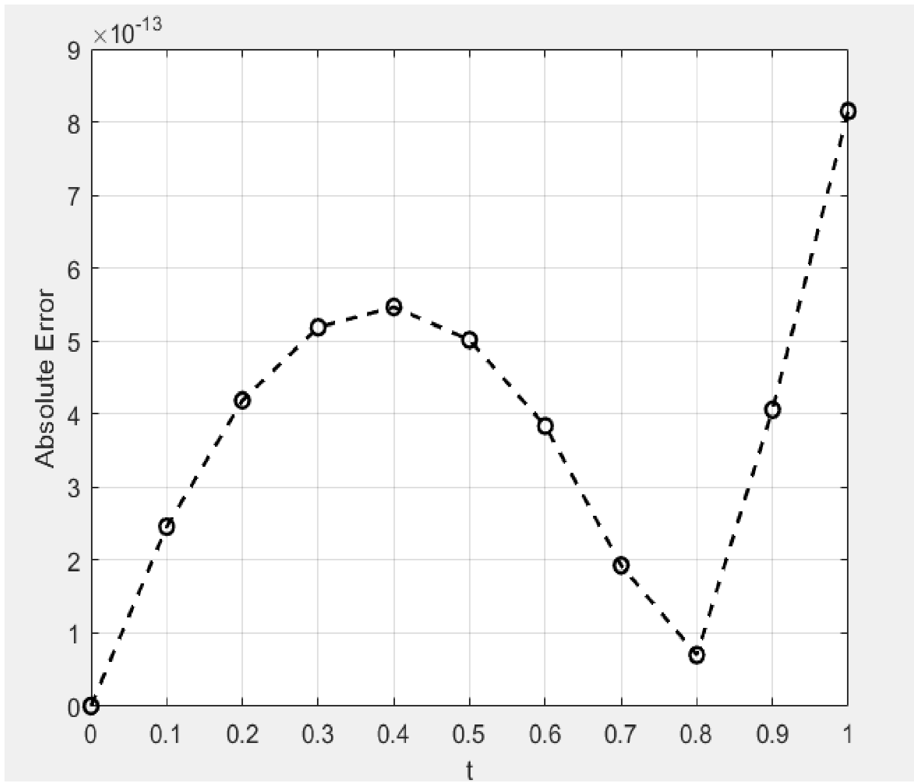


Figure 4.7: Absolute Error Between Exact and Approximate Solutions for equation (4.27) at $M = 3$

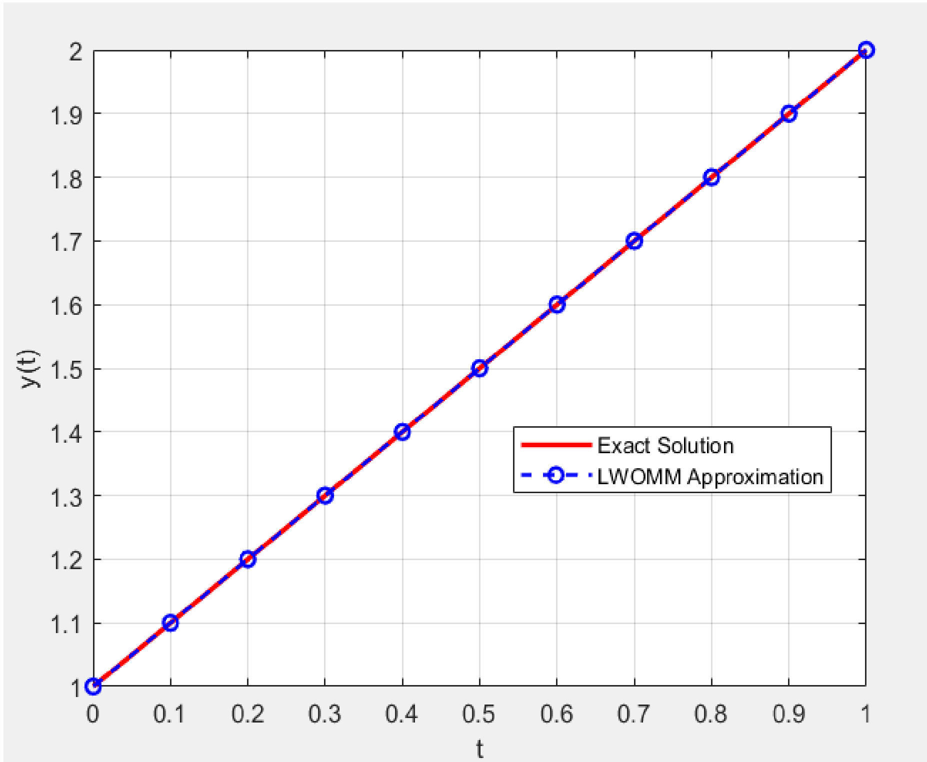


Figure 4.8: Comparison Between the Exact and the Approximate Solutions Using the Fractional Legendre Wavelet Operational Matrix for $M = 3$

This example 4, was solved using the Legendre wavelet operational matrix method **(LWOM)** for $M = 3$. The numerical results and the comparison between the exact and approximate solutions demonstrated excellent agreement as shown in Table (4.6) and Figures (4.7) and (4.8). This confirms the reliability and efficiency of the proposed fractional approach, even with a relatively small value of M. The success of this example adds further evidence to the accuracy and applicability of the method in solving fractional differential equations.

Chapter Five

Conclusion and Future Worke

5.1. Conclusion

In this research, I presented an overview of fractional calculus, including special functions such as the Beta and Gamma functions. I also discussed several important definitions, with a particular focus on the Riemann–Liouville definition, presenting its key properties, proofs, and examples. Additionally, I introduced the Caputo definition, which I adopted throughout my thesis, and highlighted its properties, proofs, and illustrative examples.

In the second chapter, I explored the Legendre polynomials, their generating functions, orthogonality, and provided various examples. The third chapter focused on wavelets: I defined wavelets, reviewed their historical development and key properties, and presented some of the most well-known wavelets such as the Haar and Mexican Hat wavelets, accompanied by illustrative graphs. I also discussed Legendre and shifted Legendre wavelets, clarified the differences between them, and provided relevant proofs.

The fourth chapter focused on the numerical solution of fractional differential equations using the operational matrix approach based on the Caputo derivative. I presented the method for constructing the normalized shifted Legendre operational matrix of fractional derivatives and applied it to solve linear fractional differential equations. Four examples were solved:

The first example was taken from a published paper. I solved and obtained excellent results. It is worth noting that the paper contained an error in the matrix, which I corrected in my work.

The second example, also from a research paper, was solved and again, the results were highly accurate.

The third and Fourth examples was based on my own formulation. I achieved highly accurate results using MATLAB, which was employed in solving all examples in this chapter. The use of shifted Legendre wavelets in this study proved to be effective and produced precise numerical,

5.2. Future Work

Based on the accurate results obtained in this study, future research may focus on extending the application of Shifted Legendre Wavelets to solve nonlinear fractional differential equations and exploring the impact of varying parameters on solution accuracy. It is also recommended to investigate the use of other types of wavelets and compare their numerical efficiency with Legendre wavelets. Furthermore, the proposed method could be developed to address variable-order fractional differential equations, which reflect more realistic models in physical and engineering applications.

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Appendix A

The MATLAB Code

1) Below we show a MATLAB code for **Example 1** using Legendre wavelet operational matrix method **(LWOM)**, this code is constructed by the author of this thesis.

```
%% MATLAB Code to Solve Example 4.1 using Shifted Legendre Wavelet Operational Matrix Method
% Using the Shifted Legendre Wavelet Operational Matrix Method with M=3, k=0

% Clear workspace and figures
clear all;
close all;
clc;

%% Parameters
M = 3; % Number of wavelet terms (M=3 means indices 0,1,2,3 for a 4x4 matrix)
k = 0; % Shift parameter

%% Define constants and exact solution for comparison
pi_val = pi;
exact_solution = @(t) sqrt(pi_val * (t + 1));

%% Define the operational matrices for Shifted Legendre Polynomials
  % D matrix (4x4) for Shifted Legendre Polynomials
  D = [
      0,           0,           0,           0;
      2*sqrt(3),   0,           0,           0;
      0,           2*sqrt(15),  0,           0;
      0,           0,           2*sqrt(35),  0
  ];

  % D^(3/2) matrix (4x4) for Shifted Legendre Polynomials
  % Note: These matrices need to be recalculated for Shifted Legendre
```

```matlab
D_3_2 = [
    0,                    0,                        0,                      0;
    0,                    0,                        0,                      0;
    16*sqrt(5)/sqrt(pi_val), 16*sqrt(5)*sqrt(3)/(5*sqrt(pi_val)), -16/(7*sqrt(pi_val)), 16*sqrt(7)*sqrt(5)/(105*sqrt(pi_val));
    -80*sqrt(7)/sqrt(pi_val),-16*sqrt(7)*sqrt(3)/(7*sqrt(pi_val)), 16*sqrt(7)*sqrt(5)/(7*sqrt(pi_val)), -16/(3*sqrt(pi_val))
];

```

```matlab
% D^(5/2) matrix (4x4) for Shifted Legendre Polynomials
D_5_2 = [
    0,                    0,                        0,                      0;
    0,                    0,                        0,                      0;
    0,                    0,                        0,                      0;
    160*sqrt(7)/sqrt(pi_val), 32*sqrt(7)*sqrt(3)/sqrt(pi_val), -32*sqrt(7)*sqrt(5)/(7*sqrt(pi_val)), 32/(3*sqrt(pi_val))
];
```

```matlab
%% Time discretization for final solution evaluation
t_values = 0:0.1:1; % Discretize the time range from 0 to 1

%% Construct the system of algebraic equations
% Use more collocation points for better accuracy
num_collocation = 100; % More points for better accuracy
collocation_points = linspace(0, 1, num_collocation);

% Create the coefficient matrix and right-hand side vector
A = zeros(num_collocation, M+1);
b = zeros(num_collocation, 1);

% Fill the system using collocation method
for i = 1:length(collocation_points)
    t = collocation_points(i);

    % Right-hand side of the equation
    b(i) = sqrt(t) + sqrt(pi_val);

    % Construct the left-hand side using operational matrices
    for n = 0:M
        % Calculate psi(n,t) directly using SHIFTED Legendre polynomials
        % No transformation needed as shifted polynomials are defined on [0,1]

        % Calculate Shifted Legendre polynomial value based on n
        if n == 0
            P_val = 1;
```

```
        elseif n == 1
            P_val = 2*t - 1;
        elseif n == 2
            P_val = 6*t^2 - 6*t + 1;
        elseif n == 3
            P_val = 20*t^3 - 30*t^2 + 12*t - 1;
        end

        % Calculate psi value for Shifted Legendre
        psi_n_t = sqrt(2*n+1) * P_val;

        % Term with 1/√(t+1)
        A(i, n+1) = (1/sqrt(t+1)) * psi_n_t;

        % Apply D^(3/2) and D^(5/2) operational matrices
        for j = 0:M
            % Calculate psi(j,t) directly using SHIFTED Legendre polynomials
            if j == 0
                P_val_j = 1;
            elseif j == 1
                P_val_j = 2*t - 1;
            elseif j == 2
                P_val_j = 6*t^2 - 6*t + 1;
            elseif j == 3
                P_val_j = 20*t^3 - 30*t^2 + 12*t - 1;
            end

            psi_j_t = sqrt(2*j+1) * P_val_j;
```

```
                % Term with 4D^(3/2)
                A(i, n+1) = A(i, n+1) + 4 * D_3_2(n+1, j+1) * psi_j_t;

                % Term with 4(t+1)D^(5/2)
                A(i, n+1) = A(i, n+1) + 4 * (t+1) * D_5_2(n+1, j+1) * psi_j_t;
            end
        end
end

%% Add rows for initial conditions
% y(0) = √π
A_ic1 = zeros(1, M+1);
for n = 0:M
    % Calculate psi(n,0) directly using SHIFTED Legendre polynomials
    if n == 0
        P_val = 1;
    elseif n == 1
        P_val = 2*0 - 1; % = -1
    elseif n == 2
        P_val = 6*0^2 - 6*0 + 1; % = 1
    elseif n == 3
        P_val = 20*0^3 - 30*0^2 + 12*0 - 1; % = -1
    end

    % Calculate psi value
    psi_n_0 = sqrt(2*n+1) * P_val;

    A_ic1(1, n+1) = psi_n_0;
end
```

```
b_ic1 = sqrt(pi_val);

% dy/dt(0) = √π/2
A_ic2 = zeros(1, M+1);
for n = 0:M
    for j = 0:M
        % Calculate psi(j,0) directly using SHIFTED Legendre polyn
        if j == 0
            P_val_j = 1;
        elseif j == 1
            P_val_j = -1; % 2*0-1
        elseif j == 2
            P_val_j = 1;  % 6*0^2-6*0+1
        elseif j == 3
            P_val_j = -1; % 20*0^3-30*0^2+12*0-1
        end

        % Calculate psi value
        psi_j_0 = sqrt(2*j+1) * P_val_j;

        A_ic2(1, n+1) = A_ic2(1, n+1) + D(n+1, j+1) * psi_j_0;
    end
end
b_ic2 = sqrt(pi_val)/2;

% y(1) = √2π
A_ic3 = zeros(1, M+1);
for n = 0:M
```

```
    % Calculate psi(n,1) directly using SHIFTED Legendre polynomials
    if n == 0
        P_val = 1;
    elseif n == 1
        P_val = 2*1 - 1; % = 1
    elseif n == 2
        P_val = 6*1^2 - 6*1 + 1; % = 1
    elseif n == 3
        P_val = 20*1^3 - 30*1^2 + 12*1 - 1; % = 1
    end

    % Calculate psi value
    psi_n_1 = sqrt(2*n+1) * P_val;

    A_ic3(1, n+1) = psi_n_1;
end
b_ic3 = sqrt(2*pi_val);

% Combine the system with initial conditions
% Add weight to initial conditions for better accuracy
weight_ic = 10000; % Higher weight factor for initial conditions
A_weighted = [A; weight_ic*A_ic1; weight_ic*A_ic2; weight_ic*A_ic3];
b_weighted = [b; weight_ic*b_ic1; weight_ic*b_ic2; weight_ic*b_ic3];

% Solve the weighted system
W = A_weighted \ b_weighted;

%% Compute the approximate solution using the obtained coefficients
approx_solution = zeros(size(t_values));
for i = 1:length(t_values)
```

```
    t = t_values(i);
    for n = 0:M
        % Calculate psi(n,t) directly using SHIFTED Legendre polynomials
        if n == 0
            P_val = 1;
        elseif n == 1
            P_val = 2*t - 1;
        elseif n == 2
            P_val = 6*t^2 - 6*t + 1;
        elseif n == 3
            P_val = 20*t^3 - 30*t^2 + 12*t - 1;
        end

        % Calculate psi value
        psi_n_t = sqrt(2*n+1) * P_val;

        approx_solution(i) = approx_solution(i) + W(n+1) * psi_n_t;
    end
end

%% Calculate exact solution values
exact_values = exact_solution(t_values);

%% Create a table for comparison with the paper's results
t_table = t_values';
exact_values_table = exact_values';
approx_values_table = approx_solution';
absolute_error = abs(exact_values_table - approx_values_table);

%% Display the table
fprintf('Table 1. Comparison of our solution y_LWOMM(t) using Shifted Legendre Polynomials and the exact solution x(t) for Example 1\n');
fprintf('-----------------------------------------------------------------------\n');
fprintf('    t         y(t)           y_LWOMM(t)       Absolute error\n');
fprintf('-----------------------------------------------------------------------\n');
for i = 1:length(t_values)
    % Format the output to match the paper's table format
    if abs(absolute_error(i)) < 1e-4
        % For very small errors, use scientific notation
        fprintf('%5.1f    %12.8f    %12.8f    %12.4e\n', t_table(i), exact_values_table(i), approx_values_table(i), absolute_error(i));
    else
        % For larger errors, use fixed-point notation
        fprintf('%5.1f    %12.8f    %12.8f    %12.10f\n', t_table(i), exact_values_table(i), approx_values_table(i), absolute_error(i));
    end
end
fprintf('-----------------------------------------------------------------------\n');

```

```
%% Plot the exact and approximate solutions
figure;
plot(t_values, exact_values, 'r', 'LineWidth', 2);
hold on;
plot(t_values, approx_solution, 'b--o', 'LineWidth', 1.5);
legend('Exact Solution', 'Approximate Solution ');
xlabel('t');
ylabel('y(t)');
 %title('Comparison of Exact and Approximate Solutions for Example 1');
grid on;

%% Display the absolute error
figure;
plot(t_values, absolute_error, 'k--o', 'LineWidth', 1.5);
xlabel('t');
ylabel('Absolute Error');
%title('Absolute Error between Exact and Approximate Solutions');
grid on;
```

2) This section presents the MATLAB code used for **Example 2**.

```
%% MATLAB Code to Solve Example 4.2 using LWOMM (M = 2, k = 0)

clear all; close all; clc;
format long

%% Parameters
M = 2; % Number of wavelet terms (0 to 2 → 3x3 matrix)
k = 0; % Shift
pi_val = pi;
t_values = 0:0.1:1; % Interval [0,1] for Table 2

%% Exact solution
y_exact = @(t) (1 + t);

%% Operational Matrices (from paper for M = 2, k = 0)
D = [0, 0, 0;
     2*sqrt(3), 0, 0;
     0, 2*sqrt(15), 0];

D2 = [0, 0, 0;
      0, 0, 0;
      4*sqrt(3)*sqrt(5), 0, 0];

```

```
D32 = [0, 0, 0;
       0, 0, 0;
       16*sqrt(5)/sqrt(pi_val), (16*sqrt(5)*sqrt(3))/(5*sqrt(pi_val)), -16/(7*sqrt(pi_val))];

%% Collocation Points
N = 10;
collocation_points = linspace(0, 1, N);

%% Assemble system: D^2 y + D^{3/2}y + y = 1 + t
A = zeros(N, M+1);
b = zeros(N, 1);

for i = 1:N
    t = collocation_points(i);
    y_trans = 2*t - 1; % Transform to [0,1]

    for n = 0:M
        % psi_n(t)
        if n == 0
            Pn = 1;
        elseif n == 1
            Pn = y_trans;
        elseif n == 2
            Pn = (3*y_trans^2 - 1)/2;
        end
        psi_n = sqrt((2*n + 1)/2) * Pn;

        % Sum for D^2 y and D^{3/2}y
        total = psi_n;
        for j = 0:M
            % psi_j(t)
            if j == 0
                Pj = 1;
            elseif j == 1
                Pj = y_trans;
            elseif j == 2
```

```
                Pj = (3*y_trans^2 - 1)/2;
            end
            psi_j = sqrt((2*j + 1)/2) * Pj;

            total = total + D2(n+1,j+1) * psi_j + D32(n+1,j+1) * psi_j;
        end
        A(i,n+1) = total;
    end

    b(i) = 1 + t;
end

%% Initial conditions: y(0) = 1, dy/dt(0) = 1
% y(0)
 y0_row = zeros(1, M+1);
y = -1;
for n = 0:M
    if n == 0
        Pn = 1;
    elseif n == 1
        Pn = y;
    elseif n == 2
        Pn = (3*y^2 - 1)/2;
    end
    psi_n = sqrt((2*n + 1)/2) * Pn;
    y0_row(n+1) = psi_n;
```

```
% dy/dt(0)
dy0_row = zeros(1, M+1);
for n = 0:M
    total = 0;
    for j = 0:M
        if j == 0
            Pj = 1;
        elseif j == 1
            Pj = y;
        elseif j == 2
            Pj = (3*y^2 - 1)/2;
        end
        psi_j = sqrt((2*j + 1)/2) * Pj;
        total = total + D(n+1,j+1) * psi_j;
    end
    dy0_row(n+1) = total;
end

% Add weighted initial conditions
A = [A; 1e6*y0_row; 1e6*dy0_row];
b = [b; 1e6*1; 1e6*1];

%% Solve for coefficients
W = A \ b;

```

```
%% Approximate solution on t_values
approx = zeros(size(t_values));
for i = 1:length(t_values)
    t = t_values(i);
    y_trans = 2*t - 1;

    for n = 0:M
        if n == 0
            Pn = 1;
        elseif n == 1
            Pn = y_trans;
        elseif n == 2
            Pn = (3*y_trans^2 - 1)/2;
        end
        psi_n = sqrt((2*n + 1)/2) * Pn;
        approx(i) = approx(i) + W(n+1) * psi_n;
    end
end

%% Output Table
exact_vals = y_exact(t_values)';
abs_err = abs(exact_vals - approx');
```

```matlab
%% Output Table
exact_vals = y_exact(t_values)';
abs_err = abs(exact_vals - approx');

fprintf('\nTable 2. Comparison of our solution y_LWOMM(t) and the exact solution y(t) for Example 2\n');
fprintf('------------------------------------------------------------\n');
fprintf('   t       y(t)          y_LWOMM(t)      Absolute error\n');
fprintf('------------------------------------------------------------\n');
for i = 1:length(t_values)
    fprintf('%4.1f   %12.9f   %12.9f   %8.1e\n', t_values(i), exact_vals(i), approx(i), abs_err(i));
end
fprintf('------------------------------------------------------------\n');

%% Plot: Exact vs Approx
figure;
plot(t_values, exact_vals, 'r-', 'LineWidth', 2); hold on;
plot(t_values, approx, 'bo--', 'LineWidth', 1.5);
legend('Exact Solution', 'LWOMM Approximation', 'Location', 'Best');
xlabel('t'); ylabel('y(t)');
%title('Comparison of Exact vs LWOMM Approximate Solution');
grid on;

%% Plot: Absolute Error
figure;
plot(t_values, abs_err, 'k--o', 'LineWidth', 1.5);
xlabel('t'); ylabel('Absolute Error');
%title('Absolute Error between Exact and LWOMM Approximation');
grid on;
```

وزارة التعليم العالي والبحث العلمي
جـامعـــــة ســـــــــرت
إدارة الدراسات العليا والتدريب
كـليـــــة العلوم
قسم الرياضيات
البرنامج الماجستير

رسالة ماجستير بعنوان

(حل المعادلات التفاضلية ذات الرتبة الكسرية باستخدام مصفوفة العمليات لمويجيات متعددات الحدود– دراسة تحليلية)

قدمت هذه الرسالة كأحد متطلبات الحصول على درجة الإجازة العالية (الماجستير)

إعداد الطالبـــة: سعاد مفتاح عمر أبوخزرانة

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المُلخَص

المعادلات التفاضلية الكسرية تُعمّم المشتقات والتكاملات الكلاسيكية باستخدام مؤثرات المشتقة الكسـرية. وقد اكتسـبت هذه المعادلات شـعبية متزايدة في السـنوات الأخيرة نظرًا لتطبيقاتها في مجالات متنوعة، بما في ذلك الفيزياء والهندسة والعلوم المالية.
تختص هذه الرسالة بدراسة طريقة عددية لحل المعادلات التفاضلية ذات الرتبة الكسرية الخاضعة لشـروط ابتدائية أو حدية باسـتخدام تحسـين مويجيات ليجندر. وتتمثل الفكرة الأسـاسـية لهذه الطريقة في تقريب كل من الدوال المعطاة والمجهولة داخل المعادلة المراد حلها باستخدام مويجات ليجندر. وبالاستفادة من خاصية التعامد لمتعددات حدود ليجندر ومصفوفة العمليات للمشتقات العادية والكسرية (وفقًا لتعريف كابوتو للمشتقة الكسرية)، يتم اختزال المعادلة التفاضلية الكسرية إلى نظام من المعادلات الجبرية التي يمكن حلها بسـهولة. للتحقق من دقة وكفاءة هذه الطريقة، تم تقديم عدد من الأمثلة التوضيحية لمسائل القيم الابتدائية والحدية. أيضاً تم مقارنة النتائج العددية التي تم الحصـول عليها بالحلول المضـبوطة (التحليلية)، مما يؤكد فعالية هذه الطريقة في حل المعادلات التفاضلية ذات الرتبة الكسرية.

الكلمات المفتاحية: الحساب الكسري- مويجات ليجندر- مصفوفة العمليات- التفاضل الكسري- مشتقة كابوتو الكسرية.