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Title

Fractional Sumudu Transform and its relation to some known Integral Transforms and some Applications.

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Declaration

I, the student **Entesar Abdulhadi Abdulaziz Mohammed**, registered at Sirte University, Faculty of Science, hereby declare that I have prepared this dissertation myself, entitled (**Fractional Sumudu Transform and its relation to some known Integral Transforms and some Applications**) and that I have relied only on the sources and references documented in the body of the dissertation and the reference list. No part of this work has been copied or quoted without proper citation.

I also confirm that this work has not been submitted to any other institution or party for the purpose of obtaining an academic or scientific degree.

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Abstract

This dissertation investigates the fractional Sumudu transform, offering a comprehensive analysis of its properties and applications. Also defining the transform through both Riemann-Liouville and Caputo fractional derivatives, the study establishes a theoretical foundation for its use. Key properties, including scale preservation, shifting theorem, and the convolution theorem, are rigorously examined. Furthermore, the dissertation elucidates the relationships between the Sumudu transform and other established integral transforms, such as Laplace, Fourier, and Mellin transforms, providing a comparative perspective. The inherent advantage of the Sumudu transform, its ability to solve differential equations without altering the units or distorting the shape of the original function, is emphasized and demonstrated. Some practical applications are presented, showcasing the transform's efficacy in solving fractional-order differential equations. Additionally, the dissertation explores the utility of the standard Sumudu transform in modeling real-world phenomena, specifically by analyzing population growth dynamics and determining production and loss rates for products. This research contributes significantly to the understanding and application of the Sumudu transform in various scientific and engineering contexts.

Dedication

To those who planted in my heart the love of knowledge and faith, to those who raised their hands in prayer for me at all times…

... To my dear parents…

To the flower of my life and the joy of my days…you were a great support to me to continue despite the fatigue… your smiles gave me strength and hope …

... To my dear children…

To my companion on the path… Who accompanied me through the hardships of the road … whose support and backing were the reason for continuing the journey…

… To my dear husband…

To all my loved ones and friends...to those whose prayers preceded their own existence…
I dedicate this research to you with deep gratitude and love …

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Praise and extend be to Allah who has granted us the joy of the final destination after He has instilled in us the certainty of arrival

I extend firstly my special thanks to the doctor supervising this research…

... **Prof. Dr. Haniyah Saed**... For her continuous support and guidance... Until we brought this research into existence...

I extend also to everyone who helped me with even a word... To everyone who taught me even a letter... To all the faculty members in the department...

I also thank my family for their constant support and all my friends and colleagues and everyone who gave me hope and helped me persevere until arrived …

Table of contents

List of Symbols.

Symbol	Definition
$F(s)$	Laplace transform of frequency variable s.
$\mathcal{F}$	Fourier transform.
$\mathcal{H}_n$	Hankel transform.
$\mathcal{M}$	Mellin transform.
$G(v)$	Sumudu transform of complex variable v.
I^{α}	Fractional integral operator.
D^{α}	Fractional differential operator.
$D_t^q \varphi(t)$	Riemann-Liouville fractional derivative.
${}_0^c D_t^{\alpha} \varphi(t)$	Caputo fractional derivative.
$\Gamma(m)$	Gamma function.
$\beta(k, n)$	Beta function.
$\varphi(t)$	Function of time t.
$E_{\alpha}(t)$	Mittag-Leffler function.
$G_{\alpha}(v)$	Fractional Sumudu integral transform.
$S_2(x, t)$	Double Sumudu transform.

Introduction

Background:

Many mathematical tools have been discovered that help in finding mathematical solutions in various fields. These tools are known as integral transforms, which have played a significant role in simplifying the solution of complex problems and transforming them into easy-to-handle algebraic problems. Among these transforms are the Laplace, Fourier, Mellin, and Hankel transforms. These transforms are known for their effect in solving ordinary and partial differential equations of integer and fractional orders, as well as linear and nonlinear types, as well as solving complex integral equations. The effect of their application has appeared in various scientific fields such as applied mathematics, physics, and engineering. Scientists have not stopped searching for other transforms that are no less important than the transforms that were previously known, which is considered a modern transforms, is Sumudu integral transform (Gupta et al., 2010). (Duran, 2020) Mohaned transform (Attaweel & Almassry, 2020), Anuj transform (Aggarwal et al., 2023), ZZ transform(Mahgoub & Alshikh, 2016), Seji transform(Mehdi et al., 2023), and Kushare transform(S. R. Kushare et al., 2021a).

The Sumudu transform was chosen as a subject of study because it has many interesting properties, such as the scale preserving property and the convolution product property.

It is said that the reason for naming this transform the Sumudu transform is in reference to the scientist Watugala Sumudu, was the first discovered it in 1993. It is said that the word Sumudu means bright face in Sinhala (one of the languages of Sri Lanka) but the reason for the name is most likely in reference to the scientist's family name.

Literature Review:

(Belgacem et al., 2003) introduced the Sumudu transform and some fundamental properties, and used the transform to solve problems without resorting to the new frequency domain. Also (Belgacem, 2004) covered some properties of the Sumudu transform and studied integral production depreciation and indefinite population dynamics problems. Moreover, (Belgacem & Karaballi, 2006) generalized all existing Sumudu differentiation integration, and convolution theorems. They introduced recurrence results and used Sumudu transform in shift theorems, introducing a paradigm shift into the thinking of transform usage, concerning solving differential equations that may be unique to this transform due to its unit–preserving properties.

(Gupta et al., 2010) established a new definition of fractional-order Sumudu transform and found a relationship between the fractional Laplace and Sumudu transform. In addition, the researchers (Eltayeb & Kılıcman, 2010) introduced some relationships between the Sumudu transform and the Laplace transform and provided some examples regarding second-order differential equations with nonconstant coefficients as a special case.

Kilicman et al.,(2010) introduced the Maxwell equations of time-fractional order in loss media and used the Sumudu transform to solve these equations.

Kilicman et al., (2011) discussed the existence of the double Sumudu transform and studied the relationship between the Laplace and Sumudu transforms. In addition, he applied these two integral transforms to solve linear ordinary differential equations.

Kiliçman & Eltayeb, (2012) studied the relationship between the Sumudu and Laplace transforms and supported the comparison with some examples.

Darzi et al., (2013) solved Cauchy problems for differential equations with the Caputo fractional derivative and solved of the fractional Diffusion-Wave equation by using the Sumudu transform techniques. Poonia (2013) discussed the infinite Sumudu transform and used the Laplace transform to solve linear ordinary differential equations with constant coefficients with convolution terms.

Karbalaie et al., (2014) presented a new algorithm to find the exact solution of a nonlinear time fractional partial differential equation, and combined the homotopy perturbation method by using the Sumudu transform.

Kılıçman & Altun, (2014) established a relationship between the fractional Laplace transform and the fractional Sumudu transform and applied a new definition to solve fractional differential equations. Furthermore, (Khalaf & Belgacem, 2014) showed the extraction of the Laplace, Fourier, and Mellin transforms than the Sumudu transform.

Demiray et al., (2015) gave the properties of Sumudu transform method (STM) and applied them to fractional-type ordinary differential equations, both homogeneous and inhomogeneous ones. In addition, (Belgacem et al., 2017) treated the transverse electromagnetic wave propagation in lossy media to get the transient electric field solution of Maxwell's equations by using the Sumudu transform.

Amer et al., (2018) found the approximate analytical solution for a system of fractional nonlinear equations of Emden-Fowler type by using the Sumudu transform method.

AL-zahour et al., (2019) generalized the formula of the Sumudu transform to the conformable fractional order and presented the general analytical solution of a singular and nonlinear conformable fractional Poisson-Boltzmann differential equation based on the conformable fractional Sumudu transform.

Saeed et al, (2020) discussed the kernel of the fractional Sumudu transform and its relationship with the fractional Fourier transform, which plays a significant role in the fractional Sumudu transform. (2020), Duran defined the degenerate Laplace transform and the degenerate Sumudu transform and established some properties and relations.

Almohmoud (2022) presented some new applications of the Sumudu transform to solve linear Volterra-type integral equations.

Problem statement:

How to apply the Sumudu transform to solve fractional-order differential equations, and what is the relationship with some other well-known transforms?

What are the most important applications of such transform for ordinary and partial differential equations containing fractional orders?

Dissertation Importance:

The importance of the study lies in how to solve fractional differential equations by using Sumudu transform that are difficult to solve analytically by normal methods.

Main objectives:

1- Study the fractional Sumudu transform, introduce some fundamental theories, and some properties.

2- Establishing the relationship between the Sumudu transform and some known integral transforms such as Laplace, Fourier, and Mellin integral transforms.

3- Showing some applications of the fractional Sumudu transform, namely solving some linear differential equations of fractional order.

Dissertation Structure:

In the **first chapter**, we will study the concept of the Sumudu transform, present some basic and important properties, and some fundamental theorems and examples.

In the **second chapter,** we will illustrate how to extract the Laplace, Fourier, and Mellin transforms from the Sumudu transform, we will introduce the relation between the Sumudu transform and other transforms, and will give some illustrative examples.

In the **third chapter**, we will study the concept of the fractional calculus, such as the definitions of the Riemann-Liouville and the Caputo fractional derivatives, we will present some of their properties and illustrative examples, and will introduce some important and special functions in fractional calculus. We will study the concept of the fractional Sumudu transform and present some of its basic properties in the sense of the Riemann-Liouville derivative and Caputo derivative.

In the **fourth chapter**, we will introduce some applications of the Sumudu transform in solving some ordinary and partial differential equations, Also, we will present solutions to some fractional-order differential equations.

In the **fifth chapter**, we will present the conclusions of reached through our study. Also, we will explain the future plan to study some advanced topics for converting fractional Sumudu transform in multiple fields.

Chapter one

Sumudu Integral Transform and some of its properties.

1. Sumudu Integral Transform and some of its properties.

1.1 Introduction.

Over the years, many integral transforms have been discovered in the literature, including Laplace, Fourier, Hankel, and Mellin transforms. These mathematical tools are basic in solving complex problems across different fields, such as mathematical physics, engineering science, and applied mathematics.

Laplace transform is one of the oldest and the most used integral transforms, the origin of this integral transform, can be traced back to the 18th century when P.S. Laplace (1749- 1827) developed it within the framework of probability theory in 1780. It became a basic technique for solving ordinary, partial differential equations, and integral equations.

Definition 1.1: The Laplace transform of a function $\varphi(t)$ is defined for $t \geq 0$, is given by:

$$F(s) = \mathcal{L}[\varphi(t)] = \int_0^\infty \varphi(t)e^{-st}\,dt\,, \qquad Re(s) > 0. \tag{1.1}$$

Where e^{-st} is the kernel of the transform, and s is a complex number.

The French scientist Joseph Fourier (1768-1830) presented his famous thesis, published in 1822. This work introduced the modern mathematical theory of differential equations, along with Fourier series and Fourier integrals, including many applications. He also provided a series of examples. Among these developments, he gave a clear formula for the Fourier transform and its inverse, and considered the second-oldest integral transform after the Laplace transform.

Furthermore, contributed greatly to solving ordinary and partial differential equations. It also played a fundamental role in addressing integral equations, particularly in applied mathematics, control engineering, and mathematical physics.

Definition 1.2: The Fourier transform for the function $\varphi(x)$ is defined for all real values of x, is given by:

$$\mathcal{F}[\varphi(x)] = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\varphi(x)dx\ , \quad k \in R \tag{1.2}$$

The scientist Hermann Hankel, a student of J. Bernhard Riemann (1839-1873), introduced a new mathematical transform known as the Hankel transform. He demonstrated that it is possible to derive this transform from the two-dimensional Fourier transform in symmetric coordinates. The Hankel transform plays a major role in solving boundary value problems in polar and cylindrical coordinates.

Definition 1.3: The Hankel transform for a function $\varphi(t)$, of order n , is defined as:

$$\mathcal{H}_n[\varphi(t)] = \tilde{\varphi}_n(tk_i) = \int_0^a tJ_n\,(tk_i)\varphi(t)dt. \tag{1.3}$$

where $J_n(tk_i)$ is the Bessel function of the first kind of order n, and is given by:

$$J_n(tk_i) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{tk_i}{2}\right)^{2m+n}$$

At the same pace of mathematical development that accompanied the nineteenth century, the scientist Mellin (1854-1933) introduced another mathematical transform, known as the Mellin transform. This transform is no less important than the Laplace and Fourier transforms. Mellin also derived its inverse formula, which is used to solve initial and boundary value problems in ordinary and partial differential equations.

Definition 1.4: Let $\varphi(x)$ is a real valued function defined on $(0, \infty)$, then the Mellin transform of the function is defined as:

$$\mathcal{M}[\varphi(x)] = \tilde{\varphi}(p) = \int_0^{\infty} x^{p-1}\varphi(x)dx. \tag{1.4}$$

Where p is a complex number. (Debnath, & Bhatta, 2015)

The twentieth century was also full of many modern transforms, which had the same importance as the transforms mentioned above. These transforms played an important role in the mathematical sciences and control engineering. In the late twentieth century, specifically in 1993, the scientist Watugala introduced a new transform known as the Sumudu transform. This transform played a major role in solving major problems in astronomy, control engineering, and applied mathematics.

Through the Sumudu transform, the time function can be transform into a complex function using a complex variable known Sumudu variable. Additionally, this transform simplifies differential equations, whether ordinary or partial, by converting them into algebraic equations that are easier to solve using known methods.

Moreover, there are other new transforms, including Kushare, Anuj, Aboodh, Mohaned, El-Zaki transforms, and others.

The Kushare transform comes from the Fourier transform and is used to solve ordinary and partial differential equations. It has a simple mathematical structure and useful properties, making it an effective tool in this field. Also related to other transforms, such as El-Zaki and Mohaned transforms, in handling mathematical equations.

Definition 1.5: The Kushare transform is defined over the set:

$$A = \{\varphi(t) | \exists M, \tau_1, \tau_2 > 0. |\varphi(t)| < M e^{\frac{|t|}{\tau_i}}, if\ t \in (-1)^i \times [0, \infty), i = 0,1\}$$

By the formula:

$$K[\varphi(x)] = w \int_0^\infty \varphi(x) e^{-x^\alpha}\, dx = S(w), \quad x \geq 0, k_1 \leq w \leq k_2. \tag{1.5}$$

Where M must be finite, k_1, k_2 may be infinite, $w > 0$, and w can be a complex number (S. R. Kushare et al., 2021).

In addition, the Anuj transform(Rashdi, 2022) is a modern integral transform used to solve differential equations with variable coefficients. Also, it is known for its ability to handle complex mathematical problems without requiring complicated computations. This transform depended on a piecewise exponential function, making it useful for various mathematical applications.

Definition 1.6: Let $\varphi(x)$ is a piecewise continuous exponential order function, then the Anuj transform is defined as:

$$A[\varphi(x)] = u^2 \int_0^\infty e^{\frac{-x}{u}} \varphi(x)dx = \varphi(u), \;\; u > 0. \quad (1.6)$$

where u is a real parameter.

Aboodh transform(Aboodh, 2013) is a modern transform derived from the Fourier integral and used to solve ordinary and partial differential equations. This transform has the property of mathematical simplicity and effective properties in making these processes easier.

Definition 1.7: The Aboodh transform is defined over the set:

$$A = \{\varphi(t) | \exists M, K_1, K_2 > 0. |\varphi(t)| < Me^{\frac{|t|}{\tau_i}}\}$$

By the formula:

$$A[\varphi(x)] = \frac{1}{w}\int_0^\infty \varphi(x)e^{-wx}\,dx = K(w), \quad x \geq 0, K_1 \leq w \leq K_2. \quad (1.7)$$

Mohaned transform (Verme & Verme, 2023) is a newly developed transform, defined by its exponential order, and used to solve linear and nonlinear differential equations by changing them into algebraic equations. This transform has many applications, particularly in physics, where it is used to solve Newton's law of Cooling, and different problems in science and engineering.

Definition 1.8: If the function $\varphi(x), x \geq 0$ is having an exponential order and is a piecewise continuous function on any interval, then the Mohand transform of $\varphi(x)$ is given

$$M[\varphi(x)] = w^2 \int_0^\infty \varphi(x)e^{-wx}dx, \;\; w \in [k_1, k_2]. \quad (1.8)$$

Among these, the El-Zaki transform (Elzaki & Elzaki, 2011) originated differently from the Sumudu transform. It is one of the methods used to solve systems of ordinary differential equations, demonstrating its efficiency in this field. Moreover, it has been widely applied in science, engineering, and astronomy.

Definition 1.9: Let $\varphi(t)$ be a function, then the El-Zaki transform is defined as:

$$E[\varphi(t)] = w\int_{0}^{\infty} \varphi(t)e^{\frac{-t}{w}}\,du = T(w), \;\; w \in (-l_1, l_2), l_1, l_2 > 0. \qquad (1.9)$$

Furthermore, to the transforms previously mentioned, numerous modern transforms were discovered recently, such as the Sara transform(Patil, 2024). The researcher introduced a solution for some problems in physics and the derivative of some functions by using Sara transform, ZZ transform(Sarah Th. Alaraji, 2025). Sara transform was used to solve Volterra equations of the second type.

Also, the Seji transform(Mehdi et al., 2023), as presented in this paper the definition of SEJI transform and some examples for solving the different initial value problems, and the Natural transform(Al-Omari & Araci, 2021). They introduced the definition of the generalized Natural transform, some properties, species of Bohemian's, the convolution theorem, convolution products, and described them in the form of auxiliary results. These transforms help to expand the use of integral transforms in solving differential equations and improving their applications in different mathematical and engineering fields.

1.2 Sumudu Integral Transform.

Definition 1.10: The Sumudu transform is defined over the set:

$$B = \{\varphi(t) | \exists M, \tau_1, \tau_2 > 0. |\varphi(t)| < Me^{\frac{|t|}{\tau_i}}, if\ t \in (-1)^i \times [0, \infty), i = 0,1\}$$

By the formula

$$G(v) = S[\varphi(t)] = \int_0^{\infty} \varphi(vt)\ e^{-t}dt\,, v \in (\,-\tau_1\,, \tau_2\,). \qquad (1.12)$$

Equivalently:

$$G(v) = S[\varphi(t)] = \frac{1}{v}\int_{0}^{\infty} \varphi(t)e^{\frac{-t}{v}}dt, v \in (-\tau_1, \tau_2). \qquad (1,13)$$

Where M must be finite, τ_1, τ_2 may be infinite, $v > 0$, and v can be a complex number. (F. B. M. Belgacem et al., 2003)

1.3 Existence of the Sumudu Integral Transform.

Theorem 1.1: (Poonia, 2013)

If $\varphi(t)$ is of exponential order, then its Sumudu transform$S[\varphi(t)] = G(v)$ is given by

$$G(v) = S[\varphi(t)] = \frac{1}{v}\int_{0}^{\infty} \varphi(t)e^{\frac{-t}{v}}dt, \qquad v \in (-\tau_1, \tau_2).$$

Where $\frac{1}{v} = \frac{1}{\mu} + \frac{i}{\tau}$ The defining integral for G exists at points $\frac{1}{v} = \frac{1}{\mu} + \frac{1}{\tau}$ in right half plan $\frac{1}{\mu} > \frac{1}{k}$ and $\frac{1}{\tau} > \frac{1}{l}$

Proof:

By $\frac{1}{v} = \frac{1}{\mu} + \frac{i}{\tau}$ and we can express $G(v)$ as

$$G(v) = \int_{0}^{\infty} \varphi(t)cos\left(\frac{t}{\tau}\right)e^{\frac{-t}{\mu}}dt - i\int_{0}^{\infty} \varphi(t)sin\left(\frac{t}{\tau}\right)e^{\frac{-t}{\mu}}dt$$

Ther for values of $\frac{1}{\mu} > \frac{1}{k}$, we have

$$\int_{0}^{\infty} |\varphi(t)|\left|cos\left(\frac{t}{\tau}\right)\right|e^{\frac{-t}{\mu}}dt \leq M\int_{0}^{\infty} e^{\left(\frac{1}{k}-\frac{1}{\mu}\right)t}\varphi(t)dt \leq \left(\frac{M\mu k}{\mu - k}\right)$$

And

$$\int_{0}^{\infty} |\varphi(t)|\left|sin\left(\frac{t}{\tau}\right)\right|e^{\frac{-t}{\mu}}dt \leq M\int_{0}^{\infty} e^{\left(\frac{1}{k}-\frac{1}{\mu}\right)t}\varphi(t)dt \leq \left(\frac{M\mu k}{\mu - k}\right)$$

Which imply that the integrals defining the real and imaginary parts of exist for value of $Re\frac{1}{\mu} > \frac{1}{k}$, the derivation is complete.

1.4 Sumudu Transform of Derivative.

Theorem 1.2: (Tuluce Demiray et al., 2015)

If $G(v)$ is the Sumudu transform of $\varphi(t)$, then the Sumudu transform of its integer-order derivative with constant coefficients is given by

$$S\left[\frac{d^n\varphi(t)}{dt^n}\right] = v^{-n}\left[G(v) - \sum_{m=0}^{n-1} v^m \frac{d^n\varphi(t)}{dt^n}\right]_{t=0} \tag{1.14}$$

Proof:

The Sumudu transform $G(v)$ of the first derivative of the function $\varphi(t)$ is given by

$$S\left[\frac{d\varphi(t)}{dt}\right] = \frac{1}{v}\int_0^\infty \frac{d\varphi(t)}{dt} e^{-t/v} dt = \lim_{q\to\infty}\frac{1}{v}\int_0^q \frac{d\varphi(t)}{dt} e^{-t/v} dt,$$

Using integration by parts:

Let $\quad u = e^{-t/v} \Longrightarrow du = \frac{-1}{v} e^{-t/v}\, dv,$

$$dv = \frac{d\varphi(t)}{dt} \Longrightarrow v = \varphi(t)$$

Then

$$\lim_{q\to\infty}\frac{1}{v}\int_0^q \frac{d\varphi(t)}{dt} e^{\frac{-t}{v}}\, dt = \lim_{q\to\infty}\left[\frac{1}{v}\, e^{-t/v}\varphi(t)\Big|_0^\infty + \frac{1}{v^2}\int_0^q e^{-t/v}\varphi(t)dt\right]$$

$$= \lim_{q\to\infty}\left[\frac{1}{v}\, e^{-t/v}\varphi(t)\Big|_0^\infty + \frac{1}{v}\left(\frac{1}{v}\int_0^q e^{-t/v}\,\varphi(t)dt\right)\right]$$

$$= -\frac{1}{v}\varphi(0) + \lim_{q\to\infty}\frac{1}{v}\left(\frac{1}{v}\int_0^q e^{-t/v}\,\varphi(t)dt\right)$$

$$= \frac{G(v) - \varphi(0)}{v}. \tag{1.15}$$

Then the Sumudu integral transform of the second-order derivative is:

$$S\left[\frac{d^2\varphi(t)}{dt^2}\right] = \frac{G(v) - \varphi(0) - v\varphi'(0)}{v^2}\,. \tag{1.16}$$

In the same way, we can derive the general formula for the Sumudu transform of any n-order integer derivative:

$$S\left[\frac{d^n\varphi(t)}{dt^n}\right] = v^{-n}\left[G(v) - \sum_{m=0}^{n-1} v^m \frac{d^n\varphi(0)}{dt^n}\right].$$

Theorem 1.3: (Eltayeb & Kılıcman, 2010b and Belgacem & Karaballi, 2006)

Let $\varphi(t) \in B,$ and $G(v)$ is Sumudu transform of the function $\varphi(t)$, then the Sumudu transform of the non-constant coefficients is given by:

1. $S[t\varphi(t)] = v\dfrac{d}{dv}\big(vG(v)\big).$ (1.17)

2. $S[t\varphi'(t)] = v^2\dfrac{d}{dv}\left[\dfrac{G(v)-\varphi(0)}{v}\right] + v\left[\dfrac{G(v)-\varphi(0)}{v}\right].$ (1.18)

3. $S[t\varphi''(t)] = v^2\dfrac{d}{dv}\left[\dfrac{G(v)-\varphi(0)-\varphi'(0)}{v^2}\right] + \left[\dfrac{G(v)-\varphi(0)-\varphi'(0)}{v^2}\right].$ (1.19)

4. $S[t^2\varphi''(t)] = v^4\dfrac{d^2}{dv^2}\left[\dfrac{G(v)-\varphi(0)-\varphi'(0)}{v^2}\right]$

$$+4v^3\frac{d}{dv}\left[\frac{G(v)-\varphi(0)-\varphi'(0)}{v^2}\right] + 2v^2\left[\frac{G(v)-\varphi(0)-\varphi'(0)}{v^2}\right]. \quad (1.20)$$

5. $S[t^n\varphi^n(t)] = v^nG(v).$ (1.21)

Proof:

1. $S[t\varphi(t)] = \displaystyle\int_0^\infty (vt)\varphi(vt)e^{-t}dt$

$$= v\int_0^\infty \frac{d}{dt}(t\varphi(vt))e^{-t}dt - vt\varphi(vt)e^{-t}|_0^\infty$$

$$= v\int_0^\infty \frac{d}{dt}t\varphi(vt)e^{-t}dt = v\int_0^\infty (\varphi(vt) + vt\varphi'(vt))e^{-t}dt$$

$$S[t\varphi(t)] = v\left[G(v) + v\frac{d}{dv}G(v)\right] = \frac{vd[vG(v)]}{dv}$$

2. $\dfrac{d}{dv}\left[\dfrac{G(v)-\varphi(0)}{v}\right] = \dfrac{d}{dv}\displaystyle\int_0^\infty \frac{1}{v}\varphi'(t)e^{-t/v}dt = \int_0^\infty \frac{d}{dv}\left[\frac{1}{v}e^{-t/v}\right]\varphi'(t)dt$

$$= \frac{1}{v^3}\int_0^\infty t\varphi'(t)e^{-t/v}\,dt - \frac{1}{v^2}\int_0^\infty \varphi'(t)e^{-t/v}dt$$

$$= \frac{1}{v^2}S[t\varphi'(t)] - \frac{1}{v}S[\varphi'(t)]$$

$$S[t\varphi'(t)] = v^2 \frac{d}{dv}\left[\frac{G(v) - \varphi(0)}{v}\right] + v\left[\frac{G(v) - \varphi(0)}{v}\right]$$

Similarly, we can prove the properties (3) and (4) by the same steps as (2).

5. $G_n(v) = \int_0^\infty \frac{d^n}{dv^n}\varphi(vt)e^{-t}dt, \quad n = 1,2,3,\ldots$

$$= \int_0^\infty t^n\, \varphi^n(vt)e^{-t}dt,$$

$$= \frac{1}{v^n}\left[\int_0^\infty (vt)^n\varphi^{(n)}\,(vt)e^{-t}dt\right] = \frac{1}{v^n}S\left[t^n\varphi^{(n)}(t)\right]$$

$$G_n(v) = \frac{1}{v^n}S\left[t^n\varphi^{(n)}(t)\right]$$

$$S\left[t^n\varphi^{(n)}(t)\right] = v^nG_n(v).$$

This theorem helps in solving differential equations with variable coefficients, making it useful in analyzing dynamic systems in physics and engineering, and applications in electrical engineering and signal processing.

1.5 Inverse Sumudu Transform.

Definition 1.11: If $G(v)$ is the Sumudu transform of a function $\varphi(t)$, then the inverse Sumudu transform is the operation that recovers the original function from its transform and is defined mathematically by:

$$\varphi(t) = S^{-1}[G(v)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} G(v)e^{t/v}dv, \tag{1.22}$$

There are several methods for finding the inverse Sumudu transform, the most common being the use of standard transform tables, or decomposing the function into partial fractions to easily obtain the inverse of each term. Operational properties such as shifting, differentiation, and integration can also be applied, while in the general case, the direct contour integral formula (Bromwich path) in the complex plane may be used.

Theorem 1.4:(Poonia, 2013)

Let $G(v)$ and $F(v)$ be the Sumudu transform of continuous functions $\varphi(t)$ and $h(t)$, respectively. If $F(v) = G(v)$, then $\varphi(t) = h(t)$ where v is a complex number.

Proof: If c is large enough, then the integral representation of φ by

$$\varphi(t) = S^{-1}[F(v)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(v)e^{t/v}dv,$$

Since $F(v) = G(v)$ is almost everywhere, then:

$$\varphi(t) = S^{-1}[G(v)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} G(v)e^{t/v}\frac{dv}{v},$$

Using the Laplace transform.

$$F(s) = \int_{-\infty}^{\infty} e^{-st}\,\varphi(t)dt,$$

But $z = st \Longrightarrow dz = sdt$

$$F(s) = \int_{-\infty}^{\infty} \varphi(\frac{z}{s})e^{-z}\frac{dz}{s},$$

Let $v = \frac{1}{s}$

$$\varphi(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} G\left(\frac{1}{s}\right)e^{-st}\frac{ds}{s} = h(t).$$

Theorem 1.5:(F. B. M. Belgacem et al., 2003)

Let $\varphi(t) \in B$ and suppose that either $\lim_{t\to o}\varphi(t)$ or $\lim_{t\to\infty}\varphi(t)$ exists. Then:

$$\lim_{v\to 0} G(v) = \lim_{t\to 0}\varphi(t), \tag{1.23}$$

$$\lim_{v\to\infty} G(v) = \lim_{t\to\infty}\varphi(t). \tag{1.24}$$

Proof:

The first limit can be obtained as,

$$\lim_{v\to 0} G(v) = \lim_{v\to o} \int_0^\infty \varphi(vt)e^{-t}dt = \int_0^\infty \left[\lim_{v\to 0} \varphi(vt)\right] e^{-t}dt$$

Let $z = vt,$

$$= \int_0^\infty \left[\lim_{z\to 0} \varphi(z)\right] e^{-t}dt = \lim_{z\to 0} \varphi(z) \int_0^\infty e^{-t}dt = \lim_{z\to 0} \varphi(z)$$

In the same way,

$$\lim_{v\to\infty} G(v) = \int_0^\infty \left[\lim_{v\to\infty} \varphi(vt)\right] e^{-t}dt = \int_0^\infty \left[\lim_{v\to\infty} \varphi(z)\right] e^{-t}dt = \lim_{z\to\infty} \varphi(z)$$

A similar argument yields the negative counterpart to (1. 24)

$$\lim_{v\to-\infty} G(v) = \lim_{t\to-\infty} \varphi(t) \qquad (1.25)$$

When the right-hand side of (1.25) exists.

This theorem is known as the **initial** and **final value theorem**.

Corollary 1.4:(Belgacem & Karaballi, 2006)

Let $\varphi(t) \in B$, then the relation between the Laplace transform $F(s)$ and the Sumudu transform $G(v)$ is given by

$$(i)\ \ G(v) = \frac{1}{v} F\left(\frac{1}{v}\right), \qquad (1.26)$$

$$(ii)\ \ F(s) = \frac{1}{s} G\left(\frac{1}{s}\right). \qquad (1.27)$$

Theorem 1.6:(F. B. M. Belgacem et al., 2003)

Let $\varphi(t) \in B, \varphi(t)$ is a **T-periodic function**, then the Sumudu transform of the function $\varphi(t)$ is given by the formula:

$$S\big(\varphi(t)\big) = \frac{\int_0^{T/v} \varphi(vt)e^{-t}dt}{1 - e^{-T/v}}. \qquad (1.28)$$

Proof: the Laplace transform of a **T-periodic function** $\varphi(t)$ is given by:

$$\mathcal{L}[\varphi(t)] = \frac{\int_0^T e^{-st}\varphi(t)dt}{1 - e^{-st}}$$

Note that

$$\int_0^T e^{-st}\varphi(t)dt = v\int_0^{T/v} e^{-t}\varphi(vt)dt$$

and that in particular,

$$1 - e^{-T/v} = \int_0^{T/v} e^{-t}\, dt,$$

By using the relation in (1.26) and (1.27),

$$G(v) = \frac{1}{v}[\mathcal{L}(\varphi(t))]_{s=\frac{1}{v}} = \frac{\int_0^T e^{-t/v}\varphi(t)dt}{v\left[1 - e^{-T/v}\right]} = \frac{\int_0^{T/v} e^{-t}\varphi(vt)dt}{1 - e^{-T/v}}$$

For example, $\varphi(t) = \sin(t)$, the theorem (1.6) predicts that

$$S(\sin(t)) = \frac{\int_0^{2\pi/v} e^{-t}\sin(vt)\, dt}{1 - e^{-2\pi/v}}$$

$$\int_0^{2\pi/v} e^{-t}\sin(vt)\, dt = \frac{v[1 - e^{-2\pi/v}]}{1 + v^2}$$

If $v = 2\pi$, then

$$\int_0^1 e^{-t}\sin(2\pi t)\, dt = \frac{2\pi[1 - e^{-1}]}{1 + 4\pi^2}. \tag{1.29}$$

Theorem 1.7:(Belgacem & Karaballi, 2006)

Let $\varphi(t)$ be the power series function, and equal:

$$\varphi(t) = \sum_{m=0}^{\infty} a_m t^m, \tag{1.30}$$

By using the Sumudu transform of the function $\varphi(t)$, then

$$G(v) = \sum_{m=0}^{\infty} m!\, a_m v^m. \tag{1.31}$$

Proof: Replace $\varphi(t)$ by its Taylor's function expansion, we obtain

$$S[\varphi(t)] = \int_0^\infty \sum_{m=0}^{\infty} \frac{\varphi^m(0)}{m!} (vt)^m e^{-t} dt = \sum_{m=0}^{\infty} \frac{\varphi^m(0)}{m!} v^m \int_0^\infty t^m e^{-t} dt$$

$$= \sum_{m=0}^{\infty} \frac{\varphi^m(0)}{m!} v^m \Gamma(m+1)$$

$$= \sum_{m=0}^{\infty} \varphi^m(0) v^m$$

$$S[(1+t)^l] = S \sum_{m=0}^{l} \frac{l!}{m!\,(l-m)!} v^m = \sum_{m=0}^{l} \frac{l!}{(l-m)!} v^m = \sum_{m=0}^{l} p_m^l v^m.$$

The result denotes that the Sumudu transform converts combinations C_m^l into permutations P_m^l, which may increase the complexity of discrete systems.

Furthermore, for $S[\varphi(t)]$ to converge within an interval containing $v = 0$, must be satisfied the conditions:

$$(1)\ \ \varphi^m(0) \to 0\ \ as\ m \to \infty.$$

$$(2)\ \ \lim_{m\to\infty} \left| \frac{\varphi^{(m+1)}(0)}{\varphi^{(m)}(0)} \right| < 1.$$

This means that the radius of convergence r of the Sumudu transform depends on that r the convergence radius of $S[\varphi(t)]$ depends on the sequenceof the function's derivatives at zero, given by:

$$r = \lim_{m\to\infty} \left| \frac{\varphi^{(m+1)}(0)}{\varphi^{(m)}(0)} \right|$$

Thus, can be used the Sumudu transform effectively in signal processing and as a detection tool, particularly in scenarios where the original signal a decreasing power tail.

Collarly1.2: (F. B. M. Belgacem et al., 2003)

Let $H(t-b)$ be the Heaviside step function, and defined as:

$$H(t-b) = \begin{cases} 0, & if\ t < b, \\ 1, & if\ t > b. \end{cases} \tag{1.32}$$

$\delta(t-b)$, it is called the Dirac delta function, where $\delta(t) = \int_0^t \varphi(t)dt$, the Sumudu transform of both functions are given by:

(i) $S[H(t-b)] = e^{-b/v}$. (1.33)

(ii) $S[\delta(t-b)] = \frac{e^{\frac{-b}{v}}}{v}$. (1.34)

The Heaviside function is known to be the antiderivative of the Dirac delta function.

Theorem 1.8: (First Shifting Theorem)(F. B. M. Belgacem et al., 2003)

Let $\varphi(t) \in B$, then the Sumudu transform of $e^{ct}\varphi(t)$ is given by:

$$S\big(e^{ct}\varphi(t)\big) = \frac{1}{1-cv} G\left(\frac{v}{1-cv}\right). \tag{1.35}$$

Proof: The Sumudu transform is defined as:

$$S\big(e^{ct}\varphi(t)\big) = \int_0^\infty \varphi(vt)e^{cvt-t}\,dt = \int_0^\infty \varphi(vt)e^{-(1-cv)t}\,dt$$

Let

$$z = (1-cv)t \Rightarrow t = \frac{z}{1-cv}.$$

Substitution in the integral, leads to

$$S\big(e^{ct}\varphi(t)\big) = \frac{1}{1-cv}\int_0^\infty \varphi(\frac{vz}{1-cv})e^{-z}\,dz = \frac{1}{1-cv} G\left(\frac{v}{1-cv}\right).$$

Theorem 1.9: (Second Shifting Theorem) (F. B. M. Belgacem et al., 2003)

Let $\varphi(t) \in B$, and let $F(s)$ and $G(v)$ denotes to the Laplace and Sumudu transforms of $\varphi(t)$ respectively. Then, the Sumudu transform of the function:

$$h(t) = \begin{cases} \varphi(t-b), & if\ t > b, \\ 0, & if\ t < b. \end{cases} \tag{1,36}$$

is given by:

$$S\big(h(t)\big) = e^{-b/v}G(v). \tag{1.37}$$

Proof:

By applying equation (1.32), the function $h(t)$, can be written as

$$h(t) = H(t-b)\varphi(t-b) \tag{1.38}$$

The Laplace transform of the function $h(t)$, is given by:

$$\mathcal{L}\big(h(t)\big) = e^{-bs}F(s) \tag{1.39}$$

Using the relationship between Sumudu and Laplace transforms from equation (1.26), then:

$$S\big(h(t)\big) = e^{-b/v}\frac{F(1/v)}{v} = e^{-b/v}G(v). \tag{1.40}$$

Theorem 1.10: (F. B. M. Belgacem et al., 2003)

Let $\varphi(t) \in B$, and let the Sumudu transform of the function $\varphi(t)$, is $G(v)$. Then,

$$S\left(\frac{1}{t}\int_0^t \varphi(\tau)d\tau\right) = \frac{1}{v}\int_0^v G(u)du. \tag{1.41}$$

Proof: The proof can begin with the right-hand side:

$$\frac{1}{v}\int_0^v G(u)du = \frac{1}{v}\int_0^v\int_0^\infty \varphi(ut)e^{-t}dtdu = \frac{1}{v}\int_0^v e^{-t}\int_0^\infty \varphi(ut)dudt$$

Let

$$z = ut, t = \frac{z}{u},$$

then

$$\frac{1}{v}\int_0^v e^{-t}\int_0^\infty \varphi(ut)dudt = \int_0^\infty \frac{e^{-t}}{v}\int_0^v \varphi(z)\frac{dz}{t}dt$$

$$= \int_0^\infty \frac{1}{vt}e^{-t}\int_0^{vt} \varphi(z)dzdt,$$

$$= \int_0^\infty \frac{1}{vt}\left[\int \varphi(z)dz\right]e^{-t}dt = S\left(\frac{1}{t}\int_0^t \varphi(\tau)d\tau\right).$$

The above theorem shows that the average of $\varphi(t)$ over the interval $[0, t]$ is mapped, under the Sumudu transform, to the average of $G(u)$ over the interval $[0, v]$.

1.6 Some Properties of the Sumudu Transform.

There are many mathematical properties related to the Sumudu transform that help facilitate mathematical operations and shorten complex operations, but we will mention the basic properties of this important transform, as follows:

1. $S[a] = a$, a constant. (1.41)

2. $S[t] = v$. (1.42)

3. $S[e^{at}] = \dfrac{1}{1 - av}$. (1.43)

4. $S[t^m] = \dfrac{v^m}{m!}$ $, m = 1,2, \dots.$ (1.44)

5. $S[\sin(at)] = \dfrac{av}{1 + a^2v^2}$. (1.45)

6. $S[\cos(at)] = \dfrac{1}{1 + a^2v^2}$. (1.46)

7. $S[\sinh(at)] = \dfrac{av}{1 - a^2v^2}$. (1.47)

8. $S[\cosh(at)] = \dfrac{1}{1 - a^2v^2}$, (1.48)

9. Linearity property:

Let φ, h be two functions, then:

$$S[B_1\varphi(t) + B_2h(t)] = B_1\phi[v] + B_2H[v] \quad (1.49)$$

10. The convolution product property:

$$S[(\varphi * h)(t)] = v\phi(v)H(v). \quad (1,50)$$

Proof:

1. $S[a] = \dfrac{1}{v}\displaystyle\int_0^\infty ae^{\frac{-t}{v}}\,dt = \dfrac{a}{v}\left[\dfrac{e^{-\frac{t}{v}}}{\frac{-1}{v}}\right]_0^\infty = a[e^0 - e^{-\infty}] = a.$

2. $S[t] = \dfrac{1}{v}\displaystyle\int_0^\infty te^{-\frac{t}{v}}dt =$

Integration by parts:

$$S[t] = \frac{1}{v}\left[t\frac{e^{\frac{-t}{v}}}{\frac{-1}{v}}\Bigg|_0^{\infty} - \int_0^{\infty} \frac{e^{\frac{-t}{v}}}{\frac{-1}{v}} dt \right] = -\int_0^{\infty} e^{\frac{-t}{v}} dt = v[e^0 - e^{-\infty}] = v.$$

3. $S[e^{at}] = \frac{1}{v}\int_0^{\infty} e^{at}\, e^{\frac{-t}{v}} dt = \frac{1}{v}\int_0^{\infty} e^{-(\frac{1}{v}-a)t}\, dt = \frac{1}{v}\left[\frac{e^{-(\frac{1}{v}-a)t}}{-(\frac{1}{v}-a)}\right|_0^{\infty}$

$$= \frac{1}{v} * \frac{v}{1-av} = \frac{1}{1-av}$$

4. $S[t^m] = \frac{1}{v}\int_0^{\infty} t^m e^{\frac{-t}{v}}\, dt =$

Also, integration by parts:

$u = t^m \Rightarrow \quad du = mt^{m-1}, \qquad dv = e^{\frac{-t}{v}} dt \;\Rightarrow v = \frac{e^{-\frac{t}{v}}}{-\frac{1}{v}}$

Then

$$= \frac{1}{v}\left[t^m * \frac{e^{-\frac{t}{v}}}{\frac{-1}{v}}\Bigg|_0^{\infty} - \frac{m}{\frac{-1}{v}}\int_0^{\infty} t^{n-1}\, e^{\frac{-t}{v}} dt \right] = m\int_0^{\infty} t^{m-1}\, e^{\frac{-t}{v}} dt,$$

Integration by parts,

$$S[t^m] = \frac{v^m}{m!}.$$

5. $S[sint(at)] = \frac{1}{v}\int_0^{\infty} sin(at)e^{-t/v}\, dt =$

Let

$$u = sin(at) \Rightarrow du = acos(at)dt, \;\; dv = e^{\frac{-t}{v}} dt \Rightarrow v = \frac{e^{\frac{-t}{v}}}{\frac{-1}{v}}$$

Then

$$\int_0^{\infty} sin(at)e^{-t/v}\, dt = sin(at)\frac{e^{-\frac{t}{v}}}{\frac{-1}{v}}\Bigg|_0^{\infty} + \frac{a}{v}\int_0^{\infty} cos(at)e^{-t/v}\, dt,$$

$$= 0 + \frac{a}{v}\int_0^\infty cos(at)e^{-t/v}\,dt,$$

Also, integration by parts:

$$= \frac{1}{v}\left[\frac{e^{\frac{-t}{v}}}{\left(-\frac{1}{v}\right)^2 + a^2}(-\frac{1}{v}\sin(at) - \mathrm{a}\cos(at)\right|_0^\infty = \frac{av}{1 + a^2v^2}.$$

Similarly, we can prove the properties (6), (7), and (8) by the same steps as in (5). (Jasim et al., 2023)

9. $S[B_1\varphi(t) + B_2h(t)] = \frac{1}{v}\int_0^\infty [B_1\varphi(t) + B_2h(t)]\, e^{-t/v}dt,$

$$= \frac{1}{v}\int_0^\infty B_1(t)\, e^{-t/v}dt + \frac{1}{v}\int_0^\infty B_2h(t)\, e^{-t/v}dt,$$

$$= \frac{B_1}{v}\int_0^\infty \varphi(t)\, e^{-t/v}dt + \frac{B_2}{v}\int_0^\infty B_2h(t)\, e^{-t/v}dt,$$

$= B_1\phi(v) + B_2H(v)$. (F. B. M. Belgacem & Karaballi, 2006)

10. $S[(\varphi * h)(t)] = \int_0^\infty \varphi(t)h(t-\tau)d\tau,$

Laplace transform of the convolution theorem is given by:

$$\mathcal{L}\big((\varphi * h)(t)\big) = \frac{1}{v}M(s)K(s),$$

By the duality relation,

$$S(\varphi * h)(t)) = \frac{1}{v}\mathcal{L}((\varphi * h)(t))$$

And since

$$F(v) = \frac{M(1/v)}{v},\ \ G(v) = \frac{K(1/v)}{v}$$

Then the Sumudu transform of the convolution of φ *and* h is:

$$S\big((\varphi * h)(t)\big) = \frac{M(1/v) \times K(1/v)}{v},$$

$$= v\frac{M(1/v)}{v}\frac{K(1/v)}{v},$$

$$= v\phi(v)H(v).$$

In addition to the properties mentioned above, we will present Sumudu transform of some functions in the following table, facilitating their use in solving practical problems. (F. B. M. Belgacem et al., 2003)

Table 1.1 Sumudu transform of some functions.

$\varphi(t)$	$G(v) = S(\varphi(t))$
$\frac{t^{n-1}}{(n-1)!}$	v^{n-1}
$\frac{1}{(n-1)!}t^{n-1}e^{at}, n = 1,2,..$	$\frac{v^{n-1}}{(1-av)^n}$
$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	$\frac{v^3}{v^{1/2}(1-av)^{3/2}}$
$\frac{1}{a}sinh\,(at)$	$\frac{v}{1-a^2v^2}$
$\frac{1}{z}e^{at}sin\,(zt)$	$\frac{v}{(1-av)^2+z^2v^2}$
$e^{at}cos\,(zt)$	$\frac{1-av}{(1-av)^2+z^2v^2}$
$\frac{\mathrm{t}}{2\mathrm{z}}\sin\,(\mathrm{zt})$	$\frac{v^2}{(1+z^2v^2)^2}$
$\frac{1}{2z^3}(sin\,(zt) - ztcos(zt))$	$\frac{v^3}{(1+z^2v^2)^2}$
$2\sqrt{\frac{t}{\pi}}$	$\sqrt{v}$
$H(t-b)$	$e^{-b/v}$
$\delta(t-b)$	$\frac{1}{v}e^{-b/v}$
$J_0(2\sqrt{kt})$	e^{-kv}
$J_0(at)$	$\frac{1}{\sqrt{1+av^2}}$
$\frac{2}{t}(1-\cos\,(zt))$	$\frac{1}{v}ln(1+z^2v^2)$

$\varphi(t)$	$G(v) = S(\varphi(t))$
$\frac{1}{t}\sin(zt)$	$\frac{1}{v} arccot \frac{1}{v}$
$\frac{1}{2k^2} sin\,(kt) sinh\,(kt)$	$\frac{v^2}{1+4k^4v^4}$
$\frac{1}{2k^3}(sin\,(kt) - sinh\,(kt))$	$\frac{v^3}{1-k^4v^4}$
$\frac{2}{t}(1 - cosh\,(at))$	$\frac{1}{v} ln(1-a^2v^2)$
$\frac{1}{z^3}(zt - sin\,(zt))$	$\frac{v^2}{1+z^2v^2}$
$\frac{1}{(a-b)}(ae^{bt} - be^{at}), a \neq b$	$\frac{1}{(1-av)(1-bv)}$
$\frac{1}{(a-b)}(e^{at} - e^{bt}), a \neq b$	$\frac{v}{(1-av)(1-bv)}$
$\frac{1}{t}(e^{bt} - e^{at})$	$\frac{1}{v} ln \frac{1-av}{1-bv}$
$\frac{1}{b^2-a^2}(\cos(at) - \cos(bt))$, $a^2 \neq b^2$	$\frac{v^2}{(1+a^2v^2)(1+b^2v^2)}$
e^{at}	$\frac{1}{1-av}$
t^m	$\frac{v^m}{m!}$
$sin\,(at)$	$\frac{av}{1+a^2v^2}$
$cos\,(at)$	$\frac{1}{1+a^2v^2}$
$sinh(at)$	$\frac{av}{1-a^2v^2}$
$cosh(at)$	$\frac{1}{1-a^2v^2}$

Corollary 1.3: (F. B. M. Belgacem & Karaballi, 2006)

Let $h(t), g(t), f(t), f_1(t), f_2(t), \ldots\ldots,$ and $f_n(v)$, and assume they have the Sumudu transforms $H(v), G(v), F(v), F_1(v), F_2(v), \ldots\ldots,$ and $F_n(v)$, respectively. Then, the Sumudu transform of

$$(\varphi * h)^n(t) = \int \int_0^t \dots \int_o^t \varphi(\tau)h(t-\tau)(d\tau)^n \qquad (1.51)$$

is given by

$$S\big((h * g)^n(t)\big) = v^n H(v)G(v). \qquad (1.52)$$

For $n \geq 1$,

$$S[(f_1 * f_2 * \dots * f_n)(t)] = v^{n-1}F_1(v)F_2(v)\dots F_n(v). \qquad (1.53)$$

If $n = 3$, then triple convolution functions $(h * g * f)$, by the Sumudu transform is given

$$S[\,(h * g * f)(t)\,] = v^2 H(v)G(v)F(v).$$

Example1.1:(F. B. M. Belgacem & Karaballi, 2006)

Solve the following initial value problem by the Sumudu transform:

$$\frac{d\varphi}{dt} + \varphi = 1, \qquad \varphi(0) = 0.$$

Solution: By using equation (1.15), then

$$\frac{G(v) - \varphi(0)}{v} + G(v) = 1$$

$$\frac{G(v)}{v} + G(v) = 1$$

$$G(v)(1+v) = v \Longrightarrow G(v) = \frac{v}{1+v}$$

$$G(v) = 1 - \frac{1}{1+v},$$

Taking the inverse Sumudu integral transform of this equation

$$\varphi(t) = 1 - e^{-t}.$$

Example1.2:(F. B. M. Belgacem & Karaballi, 2006)

Solve the second-order ordinary differential equation:

$$\frac{d^2x(t)}{dt^2} + k^2x(t) = 0$$

Solution: Taking the Sumudu transform of this ODE,

$$\frac{G(v)-x(0)}{v^2}-\frac{x'(0)}{v}+k^2G(v)=0$$

$$G(v)-x(0)-vx'(0)+v^2k^2G(v)=0$$

$$G(v)(1+v^2k^2)=x(0)+vx'(0)$$

$$G(v)=x(0)\frac{1}{1+v^2k^2}+x'(0)\frac{v}{1+v^2k^2}$$

The inverse Sumudu transform

$$x(t)=x(0)cos(kt)+\frac{x'(0)}{k}sin(kt).$$

Example 1.3 (Eltayeb & Kılıcman, 2010b) :Solve the initial value problem:

$$\varphi''(t)-\varphi(t)=0$$

$$\varphi(0)=2,\quad \varphi'(0)=-1$$

Solution: By using the differential equations (1.16), then

$$\left[\frac{G(v)-2-v(-1)}{v^2}\right]-G(v)=0$$

$$(G(v)-2+v)-v^2\big(G(v)\big)=0$$

$$G(v)(1-v^2)=2-v$$

$$G(v)=\frac{2-v}{(1-v^2)}$$

$$G(v)=\frac{2-v}{(1-v)(1+v)}$$

By using partial fractional decomposition

$$\frac{2-v}{(1-v)(1+v)}=\frac{A}{1-v}+\frac{B}{1+v}$$

$$A=\frac{1}{2},B=\frac{3}{2}$$

Then

$$G(v)=\frac{1}{2}\frac{1}{1-v}+\frac{3}{2}\frac{1}{1+v}$$

Taking the inverse Sumudu transform,

$$\varphi(t) = \frac{1}{2}e^{t} + \frac{3}{2}e^{-t}$$

Example 1.4:(Eltayeb & Kılıcman, 2010)

Solve the differential equation:

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = \delta(t) + \frac{d^2\delta(t-1)}{dt^2} + sin(t)H(t). \qquad (1.54)$$

With the initial conditions $x(0) = 0, x'(0) = 0$

Solution:

The Sumudu transform of the function $x(t)$is given by:

$$S(x(t)) = G(v) = \int_0^\infty x(t)e^{-vt}\,dt.$$

$$S\left(\frac{d^2x}{dt^2}\right) = \frac{G(v) - x(0) - vx'(0)}{v^2},$$

$$S\left(\frac{dx}{dt}\right) = \frac{G(v) - x(0)}{v}.$$

$$S\big(\delta(t)\big) = 1, S\left(\frac{d^2\delta(t-1)}{dt^2}\right) = \frac{e^{-1/v}}{v^2}\,, S(sin\,(t)) = \frac{v}{1+v^2}$$

$$\left(\frac{G(v) - x(0) - vx'(0)}{v^2}\right) - 3\left(\frac{G(v) - x(0)}{v}\right) + 2G(v) = 1 + \frac{e^{-1/v}}{v^2} + \frac{v}{1+v^2}$$

$$G(v)\left(\frac{1}{v^2} - \frac{3}{v} + 2\right) = 1 + \frac{e^{-1/v}}{v^2} + \frac{v}{1+v^2}$$

$$G(v) = \frac{v^2}{1 - 3v + 2v^2} + \frac{e^{-1/v}}{1 - 3v + 2v^2} + \frac{v^3}{(v^2+1)(1 - 3v + 2v^2)}$$

Using Laplace transform and replace v by $\frac{1}{s}$

$$G\left(\frac{1}{s}\right) = \frac{\left(\frac{1}{s}\right)^2}{1 - 3\left(\frac{1}{s}\right) + 2\left(\frac{1}{s}\right)^2} + \frac{e^{-s}}{1 - 3\left(\frac{1}{s}\right) + 2\left(\frac{1}{s}\right)^2} + \frac{\left(\frac{1}{s}\right)^3}{\left(\left(\frac{1}{s}\right)^2 + 1\right)1 - 3\left(\frac{1}{s}\right) + 2\left(\frac{1}{s}\right)^2}$$

$$G\left(\frac{1}{s}\right) = \frac{1}{(s^2 - 3s + 2)} + \frac{s^2e^{-s}}{(s^2 - 3s + 2)} + \frac{s}{(s^2+1)(s^2 - 3s + 2)}$$

For the   term in $G\left(\frac{1}{s}\right)$, by using partial fractional decomposition,

$$\frac{1}{(s^2-3s+2)}=\frac{A}{s-1}+\frac{B}{s-2}$$

$$\frac{1}{(s^2-3s+2)}=\frac{-1}{s-1}+\frac{1}{s-2}$$

$$\mathcal{L}^{-1}\left[\frac{-1}{s-1}+\frac{1}{s-2}\right]=-e^{t}+e^{2t}$$

For the second term, by applying the Cauchy residue theorem,

$$x(t)=\mathcal{L}^{-1}[F(s)]=\sum Residues\ of(s)\ Ye^{st}\ at\ all\ poles$$

$$\mathcal{L}^{-1}\left[\frac{s^2e^{-s}}{(s-1)(s-2)}\right]=\sum Residues\ of\frac{s^2e^{-s}}{(s-1)(s-2)}$$

Poles at $s=1, s=2$, we have

$$Res_1=\lim_{s\to 1}(s-1)\frac{s^2e^{-s}e^{st}}{(s-1)(s-2)}=\frac{e^{-1}e^{t}}{-1}=-e^{t-1}$$

$$Res_2=\lim_{s\to 2}(s-2)\frac{s^2e^{-s}e^{st}}{(s-1)(s-2)}=\frac{4e^{-2}e^{2t}}{2-1}=e^{2t-2}$$

For the third term, by using partial fractional decomposition

$$\frac{s}{(s^2+1)(s^2-3s+2)}=\frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{s-2}$$

$$(s^2-3s+2)=(s-1)(s-2),$$

$$(s^2+1)=(s-i)(s+i).$$

then

$$s=(As+B)(s-1)(s-2)+C(s^2+1)(s-2)+D(s^2+1)(s-1)$$

where

$s=1$:

$1=(A(1)+B)(1-1)(1-2)+C((1)^2+1)(1-2)+D((1)^2+1)(1-1)$

$\boldsymbol{C=\frac{-1}{2}}$

$s=2$:

$$2 = (A(2) + B)(2-1)(2-2) + C((2)^2+1)(2-2) + D((2)^2+1)(2-1)$$

$$2 = 0 + 0 + 5D \Longrightarrow \boldsymbol{D} = \frac{\mathbf{2}}{\mathbf{5}}$$

$$s = 0 \Longrightarrow 0 = 2B - 2C - D \Longrightarrow 2B = 2C - D = 2\left(\frac{-1}{2}\right) + \frac{2}{5} \Longrightarrow \boldsymbol{B} = -\frac{\mathbf{3}}{\mathbf{10}}$$

$$s = i \Longrightarrow i = (Ai + B)(i-1)(i-2)$$

$$\Longrightarrow i = (3A + B) + i(A - 3B) \Longrightarrow A = -3B$$

Then

$$A - 3(-3A) = 1 \Longrightarrow A + 9A = 1 \Longrightarrow \boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{10}}$$

Substitution in third term

$$\frac{s}{(s^2+1)(s^2-3s+2)} = \frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} - \frac{\frac{1}{2}}{s-1} + \frac{\frac{2}{5}}{s-2}$$

$$\mathcal{L}^{-1}\left[\frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} - \frac{\frac{1}{2}}{s-1} + \frac{\frac{2}{5}}{s-2}\right] = \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t) - \frac{1}{2}e^t + \frac{2}{5}e^{2t}$$

Then the solution of equation (1.54) is given by

$$x(t) = \left(-e^t + e^{2t} - \frac{1}{2}e^t + \frac{2}{5}e^{2t} + \frac{1}{10}\big(cos(t) - 3sin(t)\big)\right)H(t) + (4e^{2t-2} - e^{t-1})H(t-1).$$

$$x(t) = \left(-\frac{3}{2}e^t + \frac{7}{5}e^{2t} + \frac{1}{10}\big(cos(t) - 3sin(t)\big)\right)H(t) + (4e^{2t-2} - e^{t-1})H(t-1).$$

Verification:

After solving, we verify the solution by checking the initial conditions:

$$x(0) = \left(-\frac{3}{2}e^0 + \frac{7}{5}e^{2(0)} + \frac{1}{10}\big(cos(0) - 3sin(0)\big)\right)H(0) + \big(4e^{2(0)-2} - e^{0-1}\big)H(0-1).$$

$$x(0) = -\frac{3}{2} + \frac{7}{5} + \frac{1}{10} = -\frac{15}{10} + \frac{15}{10} = 0$$

This example applies the Sumudu transform to solve a differential equation with special functions. Using the residue theorem, the time-domain solution. This method is useful in engineering and physics for analyzing dynamic systems.

1.7 Double Sumudu Integral Transform.

Definition 1.11:(Kılıcman & Gadain, 2010)

Let $\varphi(x,t); t,x \in R_+$, the function can be expressed as a convergent infinite series, then the double Sumudu transform $G(u,v)$ is given by:

$$G(u,v) = S_2[\varphi(x,t)] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\varphi(x,t)dxdt. \tag{1.55}$$

1.8 Existence Condition for the Double Sumudu Transform.

Theorem 1.11: (Bodkhe & Jagtep., 2022**)**

If $\varphi(x,t)$ is of exponential order, and let the Sumudu transform of the function $S[\varphi(x,t)] = G(u,v)$ exist, then the double Sumudu transform is given by

$$G(u,v) = S_2[\varphi(x,t)] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}f(x,t)dxdt.$$

Where

$$\frac{1}{u} = \frac{1}{\eta}+\frac{i}{\tau}, \qquad \frac{1}{v} = \frac{1}{\mu}+\frac{i}{\xi}$$

at the point $\frac{1}{u}+\frac{1}{v} = \frac{1}{\eta}+\frac{1}{\mu}+\frac{i}{\tau}+\frac{i}{\xi}$, in the half plane $\frac{1}{u}+\frac{1}{v} > \frac{1}{a}+\frac{1}{b}$.

Proof:

Using $\frac{1}{u} = \frac{1}{\eta}+\frac{i}{\tau} and \frac{1}{v} = \frac{1}{\mu}+\frac{i}{\xi}$, then

$$G(u,v) = \int_0^\infty\int_0^\infty \varphi(x,t)cos\left(\frac{x}{\tau}+\frac{t}{\xi}\right)e^{-\frac{x}{\eta}-\frac{t}{\mu}}dxdt$$

$$-i\int_0^\infty\int_0^\infty \varphi(x,t)sin\left(\frac{x}{\tau}+\frac{t}{\xi}\right)e^{-\frac{x}{\eta}-\frac{t}{\mu}}dxdt,$$

Since $\frac{1}{u}+\frac{1}{v} > \frac{1}{a}+\frac{1}{b}$, then

$$\int_0^\infty\int_0^\infty |\varphi(x,t)| \left|cos\left(\frac{x}{\tau}+\frac{t}{\xi}\right)\right| e^{-\frac{x}{\eta}-\frac{t}{\mu}}\,dxdt \leq N\int_0^\infty\int_0^\infty e^{\left(\frac{1}{a}-\frac{1}{\eta}\right)t+\left(\frac{1}{b}-\frac{1}{\mu}\right)x}dxdt$$

$$\leq N\left(\frac{\eta a}{\eta-a}\right)\left(\frac{\mu b}{\mu-b}\right)$$

And

$$\int_0^\infty\int_0^\infty |\varphi(x,t)\}| \left|sin\left(\frac{x}{\tau}+\frac{t}{\xi}\right)\right| e^{-\frac{x}{\eta}-\frac{t}{\mu}}\,dxdt \leq N\int_0^\infty\int_0^\infty e^{\left(\frac{1}{a}-\frac{1}{\eta}\right)x+\left(\frac{1}{b}-\frac{1}{\mu}\right)t}dxdt$$

$$\leq N\left(\frac{\eta a}{\eta-a}\right)\left(\frac{\mu a}{\mu-b}\right)$$

This means that the integrals of $\varphi(x,t)$ defined over the real and imaginary parts exist for the values:

$$\left(\frac{1}{u}+\frac{1}{v}\right)>\frac{1}{a}+\frac{1}{b}.$$

This theorem presents the existence condition for the double Sumudu transform.

1.9 Double Sumudu Integral Transform of Derivatives.

We will introduce some derivatives of the first-order double Sumudu transform to n partial derivatives.

Let $S_2[\varphi(x,t)] = G(u,v)$, then

1. $S_2\left[\frac{\partial\varphi(x,t)}{\partial x}\right]=\frac{G(u,v)-S[\varphi(x,0)]}{u},$ (1.56)

2. $S_2\left[\frac{\partial\varphi(x,t)}{\partial t}\right]=\frac{G(u,v)-S[\varphi(t,0)]}{v}\,,$ (1.57)

3. $S_2\left[\frac{\partial^n\varphi(x,t)}{\partial t^n}\right]=\frac{1}{v^n}G(u,v)-\sum_{i=1}^{n}\frac{1}{v^i}S\left[\frac{\partial^{n-i}\varphi(x,0)}{\partial t^{n-i}}\right],$ (1.58)

4. $S_2\left[\frac{\partial^n\varphi(x,t)}{\partial x^n}\right]=\frac{1}{u^n}G(u,v)-\sum_{i=1}^{n}\frac{1}{u^i}S\left[\frac{\partial^{n-i}\varphi(0,t)}{\partial x^{n-i}}\right].$ (1.59)

Proof:

$$\mathbf{1}.\, S_2\left[\frac{\partial\varphi(x,t)}{\partial x}\right] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\frac{\partial\varphi(x,t)}{\partial x}dxdt$$

Using integration by parts:

Let

$$u = e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} \Longrightarrow du = -\frac{1}{u}e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dx$$

$$dv = \frac{\partial\varphi(x,t)}{\partial x} \Longrightarrow v = \varphi(x,t).$$

Then the integral becomes,

$$\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\frac{\partial\varphi(x,t)}{\partial t}dxdt$$

$$= \frac{1}{uv}\left[\varphi(x,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\Big|_0^\infty - \int_0^\infty \varphi(t,x)\left(-\frac{1}{u}e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\right)dx\right],$$

$$S_2\left[\frac{\partial\varphi(x,t)}{\partial x}\right] = \frac{G(u,v) - S[\varphi(t,0)]}{u}.$$

Following the same steps, we can prove the rest of the points even n from the limits. (Jasim et al., 2023)

Theorem 1.12: (Kılıcman & Gadain, 2010)

If the double Sumudu transform of $\varphi(x,t)$ is given $G(u,v)$. Then, the double Sumudu transform of non-constant coefficients is given by:

$$\mathbf{1}.\, S_2[x^n\varphi(x,t)] = u^n\sum_{k=0}^{n} a_k^n u^k\frac{\partial^k}{\partial u^k}G(u,v), \tag{1.60}$$

$$\mathbf{2}.\, S_2[t^m\varphi(x,t)] = v^m\sum_{l=0}^{n} b_l^m v^l\frac{\partial^l}{\partial v^l}G(u,v), \tag{1.61}$$

$$\mathbf{3}.\, S_2[x^n t^m\varphi(x,t)] = u^n v^m\sum_{k=0}^{n} a_k^n\, b_l^m u^k v^l\frac{\partial^{k+l}}{\partial u^k v^l}G(u,v). \tag{1.62}$$

Where $a_0^n = 1$, $a_n^n = 1$, $a_1^n = n!\,n,\ = 1$, $a_{n-1}^n = n^2$, $a_{k-1}^{n-1} = a_{k-1}^{n-1} + (n+k)a_k^{n-1}$, similarly for b_l^m for any $k, l = 2,3, \dots, n-2$ and $l = 2,3, \dots, m-2$, $b_k^n = b_{l-1}^{m-1} + (m-l)b_l^{m-1}$. (1.63)

Proof:

The method of mathematical induction can be used for $n = 1, m = 1$ of equation (1.60) and (1.61) and similar for equation (1.62), and is true for $n = 2, m = 2$, and to perform this method to assume that the equation (1.63) applies to m and n and can prove that for equation (1.60) and (1.61) are true for $n+1, m+1$.

$$1.\ S_2[x^{n+1}\varphi(x,t)] = S_2\big[x[x^n\varphi(x,t)]\big] = u^2\frac{\partial G(u,v)}{\partial u} + uG(u,v)$$

$$= u^2\frac{\partial}{\partial u}\sum_{k=0}^{n} a_k^n u^{n+k}\frac{\partial^k}{\partial u^k}G(u,v) + u^{n+1}\sum_{k=0}^{n} a_k^n u^k\frac{\partial^k}{\partial u^k}G(u,v),$$

$$= u^2\sum_{k=0}^{n} a_k^n\left[(n+k)u^{n+k-1}\frac{\partial^k G(u,v)}{\partial u^k} + u^{n+k}\frac{\partial^{k+1}G(u,v)}{\partial u^{k+1}}\right]$$

$$+u^{n+1}\sum_{k=0}^{n} a_k^n u^k\frac{\partial^k}{\partial u^k}G(u,v)$$

$$= u^{n+1}\sum_{k=0}^{n} a_k^n\,(n+k)u^k\frac{\partial^k G(u,v)}{\partial u^k} + u^{n+1}\sum_{k=0}^{n} a_k^n u^{k+1}\frac{\partial^{k+1}G(u,v)}{\partial u^{k+1}}$$

$$+u^{n+1}\sum_{k=0}^{n} a_k^n u^k\frac{\partial^k}{\partial u^k}G(u,v)$$

$$= u^{n+1}\sum_{k=0}^{n} a_k^n\,(n+k+1)u^k\frac{\partial^k G(u,v)}{\partial u^k} + u^{n+1}\sum_{k=0}^{n} a_k^n u^{k+1}\frac{\partial^{k+1}G(u,v)}{\partial u^{k+1}}$$

$$= u^{n+1}\sum_{k=0}^{n+1} a_k^{n+1}\,u^k\frac{\partial^k G(u,v)}{\partial u^k}.$$

The remaining equation (1.61) can be proven in the same way, but as a special case when $n = 1, m = 1$, then

$$S_2[x\varphi(x,t)] = u^2\frac{\partial G(u,v)}{\partial u} + uG(u,v). \tag{1.64}$$

$$S_2[t\varphi(x,t)] = v^2 \frac{\partial G(u,v)}{\partial v} + vG(u,v). \tag{1.65}$$

Theorem1.13:(Eltayeb & Kilicman, 2010)

If the double Sumudu transform of $S_2\varphi(x,t) = G(u,v)$, then the second partial derivative of $\varphi(x,t)$, is given by:

$$\mathbf{1}. S_2\left[\frac{\partial^2\varphi(x,t)}{\partial x^2}\right] = \frac{G(u,v)}{u^2} - \frac{G(0,v)}{u^2} - \frac{1}{u}\frac{\partial G(0,v)}{\partial x}. \tag{1.66}$$

$$\mathbf{2}. S_2\left[\frac{\partial^2\varphi(x,t)}{\partial t^2}\right] = \frac{G(u,v)}{v^2} - \frac{G(u,0)}{v^2} - \frac{1}{v}\frac{\partial G(u,0)}{\partial t}. \tag{1.67}$$

Proof:

$$S_2\left[\frac{\partial^2\varphi(x,t)}{\partial x^2}\right] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\frac{\partial^2\varphi(x,t)}{\partial x^2}dxdt,$$

$$= \frac{1}{v}\int_0^\infty e^{-t/v}\left(\frac{1}{u}\int_0^\infty e^{-x/u}\frac{\partial^2\varphi(x,t)}{\partial x^2}dx\right)dt,$$

Using Integration by parts:

Let

$$u = e^{-x/u} \Rightarrow du = \frac{-1}{u}e^{-x/u}dx,$$

$$dv = \frac{\partial^2\varphi(x,t)}{\partial x^2}dx \Rightarrow v = \frac{\partial\varphi(x,t)}{\partial x}.$$

Then

$$\frac{1}{u}\int_0^\infty e^{-x/u}\frac{\partial^2\varphi(x,t)}{\partial x^2}dt = \frac{1}{u}\left[e^{-x/u}\frac{\partial\varphi(x,t)}{\partial x}\Big|_0^\infty - \int_0^\infty -\frac{1}{u}e^{-x/u}\frac{\partial\varphi(x,t)}{\partial x}\right]$$

$$\frac{1}{u}\int_0^\infty e^{-x/u}\frac{\partial^2\varphi(x,t)}{\partial x^2}dx = \frac{1}{u}\left[-\frac{\partial\varphi(x,0)}{\partial x} + \int_0^\infty \frac{1}{u}e^{-x/u}\frac{\partial\varphi(x,t)}{\partial x}\right],$$

Integration by parts, then

$$= \frac{-1}{u}\frac{\partial\varphi(0,t)}{\partial x} + \frac{G(u,v)}{u^2} - \frac{\partial\varphi(0,t)}{u\partial x}$$

Then

$$\frac{1}{v}\int_0^\infty e^{-t/v}\left(\frac{1}{u}\int_0^\infty e^{-t/u}\frac{\partial^2\varphi(x,t)}{\partial t^2}dt\right)dx=\frac{1}{v}\int_0^\infty e^{-t/v}\frac{G(u,t)}{u^2}dt$$

$$-\frac{1}{v}\int_0^\infty e^{-t/v}\frac{\varphi(0,t)}{u^2}dt-\frac{1}{u}\frac{1}{v}\int_0^\infty e^{-t/v}\frac{\partial\varphi(0,t)}{\partial x}dx.$$

$$S_2\left[\frac{\partial^2\varphi(x,t)}{\partial x^2}\right]=\frac{G(u,v)}{u^2}-\frac{G(0,v)}{u^2}-\frac{1}{u}\frac{\partial G(0,v)}{\partial x}.$$

By a similar manner, we have

$$S_2\left[\frac{\partial^2\varphi(x,t)}{\partial t^2}\right]=\frac{G(u,v)}{v^2}-\frac{G(u,0)}{v^2}-\frac{1}{v}\frac{\partial G(u,0)}{\partial t}.$$

1.10 Some Properties of the Double Sumudu Integral Transform.

1. $S_2[1]=1.$ (1.68)

2. $S_2\left[e^{\alpha x+\beta t}\right]=\dfrac{1}{(1-\alpha u)(1-\beta v)}\ ,\qquad \alpha,\beta\in R.$ (1.69)

3. $S_2[sin(\alpha x+\beta t)]=\dfrac{\alpha u+\beta v}{(1+\alpha^2u^2)(1+\beta^2v^2)}\ .$ (1.70)

4. $S_2[cos(\alpha x+\beta t)]=\dfrac{1-\alpha\beta uv}{(1+\alpha^2u^2)(1+\beta^2v^2)}\ .$ (1.71)

5. $S_2[\varphi(At)h(Bx)]=F(Au)H(Bv).$ (1.72)

Proof:

1. $S[1]=\dfrac{1}{uv}\displaystyle\int_0^\infty\int_0^\infty e^{-x/u}e^{-t/v}\,(1)dxdt=\dfrac{1}{uv}\times u.v=1.$

2. $S_2\left[e^{\alpha x+\beta t}\right]=\dfrac{1}{uv}\displaystyle\int_0^\infty\int_0^\infty e^{-x/u}e^{-t/v}\left(e^{\alpha x+\beta t}\right)dxdt$

$$=\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(\alpha-\frac{1}{u}\right)x}e^{\left(\beta-\frac{1}{v}\right)t}dxdt=\frac{1}{uv}\times\frac{u}{1-\alpha u}.\frac{v}{1-\beta v}$$

$$=\frac{1}{(1-\alpha u)(1-\beta v)}.$$

3. $S_2[sin(\alpha x+\beta t)] = \dfrac{1}{uv}\int\limits_0^\infty\int\limits_0^\infty e^{-x/u}e^{-t/v}\, sin(\alpha x+\beta t)dxdt =$

Following relation, one has

$$sin(\alpha x+\beta t) = \frac{e^{i(\alpha x+\beta t)} - e^{-i(\alpha x+\beta t)}}{2i}$$

Then

$$S_2[sin(\alpha x+\beta t)] = \frac{1}{uv}\times\frac{1}{2i}\int\limits_0^\infty\int\limits_0^\infty e^{i(\alpha x+\beta t)} - e^{-i(\alpha x+\beta t)}\, e^{-x/u}e^{-t/v}dxdt,$$

$$= \frac{1}{2iuv}\int\limits_0^\infty\int\limits_0^\infty e^{\left(i\alpha-\frac{1}{u}\right)x}e^{\left(i\beta-\frac{1}{v}\right)t}\, dxdt - \int\limits_0^\infty\int\limits_0^\infty e^{\left(-i\alpha-\frac{1}{u}\right)x}e^{\left(-i\beta-\frac{1}{v}\right)t}\, dxdt$$

$$= \frac{1}{2iuv}\left[\frac{uv}{(1-i\alpha u)(1-i\beta v)} - \frac{uv}{(1+i\alpha u)(1+i\beta v)}\right]$$

$$= \frac{1}{2i}\left[\frac{(1+i\alpha u)(1+i\beta v)-(1-i\alpha u)(1-i\beta v)}{(1+\alpha^2u^2)(1-\beta^2v^2)}\right]$$

$$= \frac{1}{2i}\left[\frac{1+i\alpha u+i\beta v-\alpha u\beta v-1+i\alpha u+i\beta v+\alpha u\beta v}{(1+\alpha^2u^2)(1-\beta^2v^2)}\right]$$

$$= \frac{1}{2i}\left[\frac{2i(\alpha u+\beta v)}{(1+\alpha^2u^2)(1-\beta^2v^2)}\right]$$

$$= \frac{\alpha u+\beta v}{(1+\alpha^2u^2)(1+\beta^2v^2)}$$

Similarly, the same steps can be used to prove property (1.71) with minor adjustments for the cosine function.

5. $S_2[\varphi(Ax)h(Bt)] = \dfrac{1}{uv}\int\limits_0^\infty\int\limits_0^\infty \varphi(Ax)h(Bt)e^{\frac{-x}{u}}e^{\frac{-t}{v}}\, dxdt,$

Let $z = Ax \Longrightarrow dz = Adx, x = \frac{z}{A}, w = Bt \Longrightarrow dw = Bdt, t = \frac{w}{B}$.

Substitution in integral

$$\frac{1}{uv}\int\limits_0^\infty\int\limits_0^\infty \varphi(Ax)h(Bt)e^{\frac{-x}{u}}e^{\frac{-t}{v}}\, dxdt = \frac{1}{uv}\int\limits_0^\infty\int\limits_0^\infty \varphi(z)h(w)e^{\frac{-z}{Au}}e^{\frac{-w}{Bv}}\frac{dz}{A}\frac{dw}{B},$$

$$= \left(\frac{1}{u} \int_0^\infty \frac{1}{A} \varphi(z)\, e^{\frac{-z}{Au}} dz \right) \left(\frac{1}{v} \int_0^\infty \frac{1}{B} h(w) e^{\frac{-w}{Bv}} dw \right),$$

$$= F(Au)H(Bv). \qquad \text{(Jasim et al., 2023)}$$

Example 1.5:(Bulut et al., 2012)

Consider homogeneous wave equation

$$u_{tt} = u_{xx}, \qquad t > 0, \quad 0 < x < \pi, \tag{1.73}$$

With initial conditions

$$u(x,0) = u_0 = \sin(x)$$

$$u_t(x,0) = 0 \tag{1.74}$$

Solution:

By applying the double Sumudu transform of derivative in equation (1.68) and

Substitution by condition, we get

$$\frac{1}{v^2} F(u,v) - \frac{1}{v^2} f(x,0) - \frac{1}{v} \frac{\partial}{\partial t} f(x,0) = F''$$

$$F'' - \frac{1}{v^2} F + \frac{1}{v^2} \sin(x) = 0$$

$$F'' - \frac{1}{v^2} F = -\frac{1}{v^2} \sin(x)$$

$$F(u,v) = c_1 e^{\frac{x}{v}} + c_2 e^{\frac{-x}{v}} \sin(x) \left[\frac{1}{1+v^2} \right]$$

$$F(u,v) = \sin(x) \left[\frac{1}{1+v^2} \right]$$

Using the inverse Sumudu transform

$$u(x,t) = \sin(x)\cos(t).$$

Chapter Two

The Relation between Sumudu Transform and some known Integral Transforms.

2. The Relation between Sumudu Integral Transform and some known Integral Transforms.

2.1 Introduction.

In this chapter, we will study the relationship of the Sumudu transform with famous transforms, such as Laplace, Fourier, and Mellin transforms, and explain how to extract these transforms from the Sumudu transform.

The Sumudu transform is known as the theoretical dual of the Laplace transform because they are similar in some properties. However, the Sumudu transform has many advantages that distinguish it from the Laplace transform and other transforms. The most important can be the solution of many problems without the need for another frequency range, as they preserve the scales, which has a beneficial effect in solving complex problems in engineering and applied sciences. It can also solve many problems easily by using Sumudu transform more than other transforms.

2.2 Laplace Transform of Derivatives.

1. $\mathcal{L}[\varphi'(t)] = sF(s) - \varphi(0).$ (2.1)

2. $\mathcal{L}[\varphi''(t)] = s^2F(s) - sF(s) - \varphi'(0).$ (2.2)

3. $\mathcal{L}\left[\varphi^{(m)}(t)\right] = s^mF(s) - \sum_{j=0}^{m-1} s^{n-j-1}\, \varphi^{(j)}(0), m = 1,2,3, \ldots$ (2.3)

(Jasim et al., 2023)

2.3 Some Properties of Laplace Transform.

1. $\mathcal{L}[1] = \frac{1}{s}.$ (2.4)

2. $\mathcal{L}[t^m] = \frac{m!}{s^{m+1}} \; , m \in N.$ (2.5)

3. $\mathcal{L}[e^{at}] = \frac{1}{s-a}, \; a \in R.$ (2.6)

4. $\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}.$ (2.7)

$$5. \mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}. \tag{2.8}$$

$$6. \mathcal{L}[\sinh(at)] = \frac{a}{s^2 - a^2}. \tag{2.9}$$

2.4 The relation between Sumudu and Laplace transforms.

In Chapter one, a direct result was given about the relationship between Sumudu and Laplace transforms given by equations (1.26) and (1.27), which will be explained by a detailed proof in the following theorem.

Theorem 2.1:(Mishra et al., 2020)

Let $\varphi(t) \in B$ and let $S[\varphi(t)] = G(k)$ and $\mathcal{L}[\varphi(t)] = F(k)$, then

$$G(k) = \frac{1}{k} F\left(\frac{1}{k}\right), \tag{2.10}$$

$$F(k) = \frac{1}{k} G\left(\frac{1}{k}\right). \tag{2.11}$$

Proof:

From $S[\varphi(t)] = G(k)$, we have

$$G(k) = \int_0^\infty \varphi(kt) e^{-t} dt, \tag{2.12}$$

Substitution

$$kt = s \Longrightarrow dt = \frac{ds}{k},$$

then

$$G(k) = \int_0^\infty \varphi(s) e^{-s/k} \frac{ds}{k}, \tag{2.13}$$

$$G(k) = \frac{1}{k} \int_0^\infty \varphi(s) e^{-s/k} ds, \tag{2.14}$$

Now $\mathcal{L}[\varphi(t)] = F(k)$, gives

$$F(k) = \int_0^\infty \varphi(t) e^{-kt}\, dt$$

Substitution in equation (2.14)

$$G(k) = \frac{1}{k} F\left(\frac{1}{k}\right)$$

Now we use $\mathcal{L}[\varphi(t)] = F(k)$,

$$\Longrightarrow \; F(k) = \int_0^{\infty} \varphi(t) e^{-kt}\, dt$$

Let $kt = s \Longrightarrow dt = \frac{ds}{k}$, then

$$F(k) = \int_0^{\infty} \varphi(\frac{s}{k}) e^{-s} \frac{ds}{k} = \frac{1}{k} \int_0^{\infty} \varphi(\frac{s}{k}) e^{-s} ds, \tag{2.15}$$

Using equation (2.12) in (2.15), then

$$F(k) = \frac{1}{k} G\left(\frac{1}{k}\right)$$

(Jasim et al., 2023)

2.5 The Differences between Laplace and Sumudu Transforms.

Sumudu transform is different from Laplace transform in that it preserves the graph of the functions and the amount of increase, decrease, and concavity i.e., the Laplace transform changes these properties, which makes studying these functions more complicated, and these differences are:

1. The First-Scale Preserving Property (Sumudu transform):(Khalaf & Belgacem, 2014)

If $S[\varphi(t)] = G(v)$, where $\varphi(t) \in B$, A is any constant, then by using definition of Sumudu transform,

$$S[\varphi(At)](v) = \int_0^{\infty} \varphi(Avt) e^{-t} dt = G(Av). \tag{2.16}$$

2. Changes of the Scale Property (Laplace transform):(Khalaf & Belgacem, 2014)

If $F(s) = \mathcal{L}[\varphi(t)]$, $\varphi(t) \in B, A$ is any constant, then

$$\mathcal{L}[\varphi(At)](s) = \int_0^{\infty} \varphi(At) e^{-st}\, dt = \frac{1}{A} F\left(\frac{s}{A}\right). \tag{2.17}$$

2.6 Illustrative examples.

Example 2.1: (Khalaf & Belgacem, 2014)

Let $\varphi(t) = t^2$,

We can apply Sumudu and Laplace transforms for this function, and using the partial integration, we obtain:

$$G(v) = S[t^2] = \frac{1}{v}\int_0^t (t^2)e^{-t/v}dt = 2v^2.$$

Whereas Laplace transform:

$$F(s) = \mathcal{L}[\varphi(t)] = \int_0^\infty (t^2)e^{-st}dt = \frac{2}{s^3}.$$

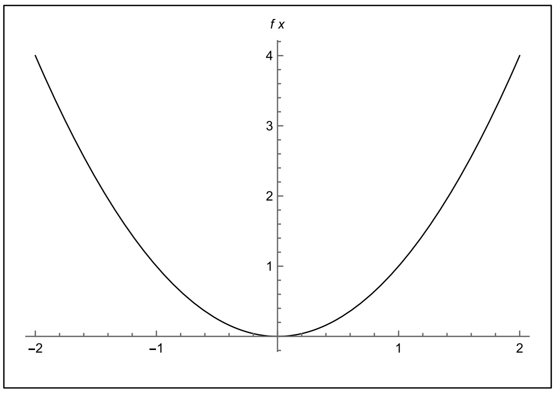

Figure 2.1, $\varphi(t) = t^2$

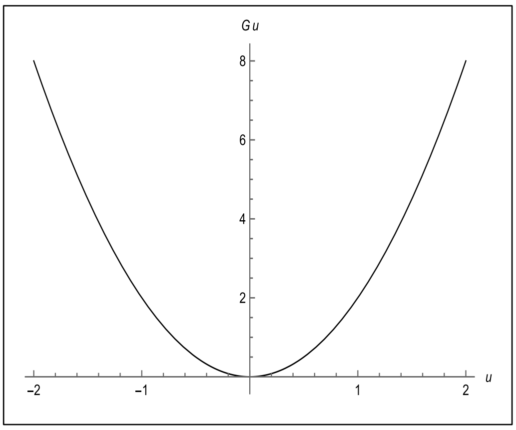

Figure 2.2, $G(v)$

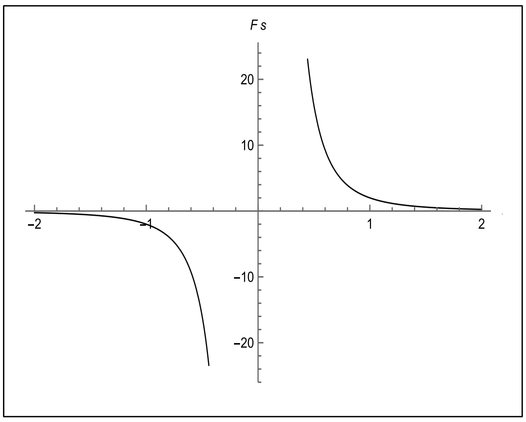

Figure 2.3, $F(s)$

When using Sumudu transform of (t) , we notice that $G(v)$, approaches the original function, unlike Laplace transform $F(s)$, which takes a different curve.

Example 2.2: (Khalaf & Belgacem, 2014)

Find Sumudu and Laplace transforms of the function

$$\varphi(t) = e^{3t},$$

The Sumudu transform can be obtained as,

$$G(v) = \frac{1}{1 - 3v}$$

The Laplace transform can be obtained as,

$$F(s) = \frac{1}{s - 3}$$

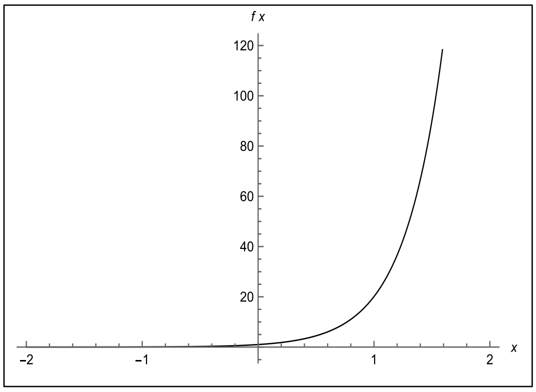

Figure 2.4, $\varphi(t) = e^{3t}$

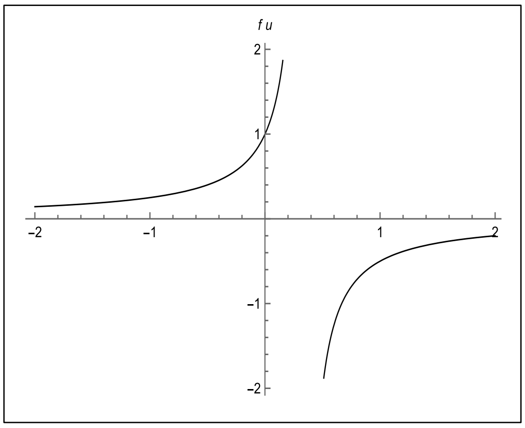

Figure 2.5, $G(v)$

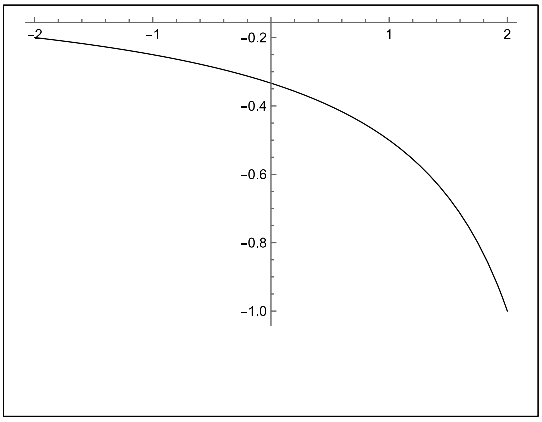

Figure 2.6, $F(s)$

In addition, to the relation between Sumudu and Laplace transforms, Sumudu transform of the Heaviside function $H(t)$ is Laplace transform of the Dirac function $\delta(t)$ and the vice versa:(F. B. M. Belgacem & Karaballi, 2006)

$$S[H(t)] = \mathcal{L}[\delta(t)] = 1. \tag{2.18}$$

$$S[\delta(t)] = \mathcal{L}[H(t)] = \frac{1}{v}. \tag{2.19}$$

We will also list some other properties of Sumudu transform that make it more distinct from Laplace transform:

- For any complex or real number A, $S[\varphi(At)] = G(Av)$.
- $S[t^m] = m!\, v^m$.
- $\lim_{t\to\infty} \varphi(t) = \lim_{v\to\tau_1} G(v)$.
- $\lim_{t\to\infty} \varphi(t) = \lim_{v\to\tau_2} G(v)$.
- $\lim_{v\to 0} \varphi(t) = \lim_{v\to 0} G(v)$.
- From Sumudu transform, definition it is easy to verify that it keeps the same units (Khalaf & Belgacem, 2014).

Example 2.3:(S. Kushare et al., 2018)

Find the solution of the system of equations.

$$\frac{dy}{dt} + x = e^t$$

$$\frac{dx}{dt} - y = -t$$

With initial conditions $x(0) = 0$ and $y(0) = 0$

Solution:

Firstly, the Solution by Sumudu transform:

$$S\left(\frac{dy}{dt}\right) + S(x) = S(e^t) \tag{2.20}$$

$$S\left(\frac{dy}{dt}\right) - S(y) = -S(t) \tag{2.21}$$

Let $S[x(t)] = X(v)$ and $S[y(t)] = Y(v)$

$$\frac{Y(v)-y(0)}{v}+X(v)=\frac{1}{1-v}$$

$$\frac{X(v)-x(0)}{v}-Y(v)=-v$$

$$\frac{Y(v)}{v}+X(v)=\frac{1}{1-v} \tag{2.22}$$

$$\frac{X(v)}{v}-Y(v)=-v \tag{2.23}$$

Then

$$Y(v)+vX(v)=\frac{v}{1-v} \tag{2.24}$$

$$X(v)-vY(v)=-v^2 \tag{2.25}$$

We solve the system (2.24) and (2.25)

$$Y(v)=\frac{v}{1-v}-vX(v) \tag{2.26}$$

Substitution in (2.25)

$$X(v)-v\left(\frac{v}{1-v}-vX(v)\right)=-v^2$$

$$X(v)-\frac{v^2}{1-v}+v^2X(v)=-v^2$$

$$X(v)(v^2+1)=\frac{v^2}{1-v}-v^2$$

$$X(v)=\frac{v^3}{(1-v)(v^2+1)} \tag{2.27}$$

Substitution by (2.27) in (2.26)

$$Y(v)=\frac{v}{1-v}-v\left(\frac{v^3}{(1-v)(v^2+1)}\right)$$

$$Y(v)=\frac{-v^4+v^3+v}{(1-v)(v^2+1)}$$

$$Y(v)=v+\frac{v^2}{(1-v)(v^2+1)} \tag{2.28}$$

By use of partial fractional decomposition:

$$X(v) = \frac{A}{(1-v)} + \frac{Bv+C}{v^2+1}$$

$$A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{2}$$

Then

$$X(v) = \frac{\frac{1}{2}}{(1-v)} - \frac{\frac{1}{2}v - \frac{1}{2}}{v^2+1}, \tag{2.29}$$

Similarly

$$Y(v) = v + \frac{\frac{1}{2}}{(1-v)} + \frac{\frac{1}{2}v - \frac{1}{2}}{v^2+1}. \tag{2.30}$$

Applying the inverse Sumudu transform

$$y(t) = S^{-1}\left(v + \frac{1}{2}\frac{1}{(1-v)} + \frac{1}{2}\frac{v}{(v^2+1)} - \frac{1}{2}\frac{1}{v^2+1}\right)$$

$$y(t) = t + \frac{1}{2}e^t + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t). \tag{2.31}$$

$$x(t) = S^{-1}\left(\frac{1}{2}\frac{1}{(1-v)} - \frac{1}{2}\frac{v}{v^2+1} - \frac{1}{2}\frac{1}{v^2+1}\right)$$

$$x(t) = \frac{1}{2}e^t - \frac{1}{2}sin(t) - \frac{1}{2}\cos(t). \tag{2.32}$$

Secondly, the solution by Laplace transform:

$$\mathcal{L}\left(\frac{dy}{dt}\right) + \mathcal{L}(x) = \mathcal{L}(e^t) \tag{2.33}$$

$$\mathcal{L}\left(\frac{dy}{dt}\right) - \mathcal{L}(y) = -\mathcal{L}(t) \tag{2.34}$$

Let $\mathcal{L}[x(t)] = X(s)$ and $\mathcal{L}[y(t)] = Y(s)$

$$sY(s) - y(0) + X(s) = \frac{1}{(s-1)} \tag{2.35}$$

$$sX(s) - x(0) - Y(s) = \frac{-1}{s^2} \tag{2.36}$$

$$sY(s) + X(s) = \frac{1}{(s-1)} \tag{2.37}$$

$$sX(s) - Y(s) = \frac{-1}{s^2} \tag{2.38}$$

By equation (2.37)

$$X(s) = \frac{1}{(s-1)} - sY(s) \tag{2.39}$$

Substitution in equation (2.38)

$$s\left(\frac{1}{(s-1)} - sY(s)\right) - Y(s) = \frac{-1}{s^2} \tag{2.40}$$

$$\frac{s}{(s-1)} - s^2Y(s) - Y(s) = \frac{-1}{s^2}$$

$$-Y(s)(s^2+1) = \frac{-1}{s^2} - \frac{s}{(s-1)}$$

$$Y(s) = \frac{1}{s^2(s^2+1)} + \frac{s}{(s-1)(s^2+1)} \tag{2.41}$$

Using fractional decomposition, one has

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$A = 0,\ \ B = 1,\ \ \ C = 0,\ \ \ D = -1$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s} - \frac{1}{s^2+1}$$

And

$$\frac{s}{(s-1)(s^2+1)} = \frac{E}{s-1} + \frac{Fs+G}{s^2+1}$$

$$E = \frac{1}{2},\quad F = -\frac{1}{2},\quad G = \frac{1}{2}$$

$$\frac{s}{(s-1)(s^2+1)} = \frac{1}{2}\frac{1}{s-1} + \frac{-{}^{s}/_{2} + {}^{1}/_{2}}{s^2+1} \tag{1.42}$$

Then

$$Y(s) = \left(\frac{1}{s} - \frac{1}{s^2+1}\right) + \left(\frac{1}{2}\frac{1}{s-1} + \frac{{-s}/{2} + {1}/{2}}{s^2+1}\right)$$

Taking the inverse Laplace transform

$$y(t) = t + \frac{1}{2}e^t - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) \tag{2.43}$$

By the same steps we get

$$x(t) = -1 + \frac{1}{2}e^t + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \tag{2.44}$$

Verification

After solving, we verify the solution by checking the initial conditions:

$$x(0) = -1 + \frac{1}{2} + \frac{1}{2} + 0 = 0, \quad y(0) = 0 + \frac{1}{2} - \frac{1}{2} - 0 = 0$$

It can be noticed through the solution using the two transforms that we get a similar solution because the Sumudu transform is very close to the Laplace transform.

2.7 The relation between double Sumudu transform and double Laplace transform.

The double Laplace transform is defined by:

$$\mathcal{L}_x\mathcal{L}_t[\varphi(x,t)] = F(p,q) = \int_0^\infty e^{-px}\int_0^\infty e^{-qt}\varphi(x,t)\,dx\,dt,$$

Where p, q are complex numbers and $x, t > 0$.

The relationship between the Sumudu transform and the Laplace transform of two variables is not much different from the relationship between Sumudu and Laplace transforms of one variable, and the following relationship illustrates this:

Let $\varphi(x,t) \in B$, with double Sumudu transform $G(u,v)$ and double Laplace transform $F(p,q)$, then

$$G(u,v) = \frac{1}{uv}\mathcal{L}_2\left(\varphi(t,x); \left(\frac{1}{u}, \frac{1}{v}\right)\right), \tag{2.45}$$

$$G(p,q) = \frac{1}{pq}\mathcal{L}_2\left(\varphi(t,x);\left(\frac{1}{p},\frac{1}{q}\right)\right), \tag{2.46}$$

This relationship between the two transforms can be best illustrated by the fact that they exchange an image $sin(x+t)$ and $cos(x+t)$:

$$S_2[sin(x+t)] = \mathcal{L}_2[\cos(x+t)] = \frac{u+v}{(1+u^2)(1+v^2)}, \tag{2.47}$$

$$S_2[cos(x+t)] = \mathcal{L}_2[\sin(x+t)] = \frac{1}{(1+u^2)(1+v^2)}, \tag{2.48}$$

Also, exchange the image of the Dirac function, $\delta(x,t) = \delta(x)\delta(t)$, and the Heaviside function, $H(x,t) = H(x) \otimes H(t)$, since

$$S_2[H(x,t)] = \mathcal{L}_2[\delta(x,t)] = 1, \tag{2.49}$$

$$S_2[\delta(x,t)] = \mathcal{L}_2[\mathrm{H}(x,t)] = \frac{1}{uv}, \tag{2.50}$$

Where the symbol $\otimes$ denotes the tensor product.

Moreover, the relation between double Laplace transform and double Sumudu transform of convolution is given by:

$$S_2[H(\varphi ** h)(x,t);u,v] = \frac{1}{uv}\mathcal{L}_2[(\varphi ** h)(x,t)]. \tag{2.51}$$

(Kiliçman & Eltayeb, 2012)

2.8 Fourier Transform of Derivative.

1. $\mathcal{F}[\varphi'(t)] = iu, \mathcal{F}[\varphi(t)] = iuF(u).$ (2.52)
2. $\mathcal{F}[\varphi^m(t)] = (iu)^m\mathcal{F}[\varphi(t)] = (iu)^m F(u).$ (2.53)

(Debnath, & Bhatta, 2015)

2.9 Some Basic Properties of Fourier Transform.

We will list some of the properties of the Fourier transform that play a major role in solving many differential and integral problems of all kinds. As we mentioned earlier, it is considered one of the oldest integral transforms.

1. (Shifting):

$$\mathcal{F}[(\varphi(t-a)] = e^{-ika}F(t). \tag{2.54}$$

2. (Conjugate):

$$\mathcal{F}\left[\overline{\varphi(-t)}\right] = \overline{\mathcal{F}[\varphi(t)]}. \tag{2.55}$$

3. (Scaling):

$$\mathcal{F}[\varphi(At \pm B) = \frac{1}{|A|} e^{\pm iBu/A} F\left(\frac{u}{A}\right),\ A \neq 0. \tag{2.56}$$

4. (Translation):

$$\mathcal{F}\left[e^{iat}\varphi(t)\right] = F(u-a). \tag{2.57}$$

5. (Duality):

$$\mathcal{F}[F(t)] = f(-u). \tag{2.58}$$

6. (Modulation):

$$\mathcal{F}[\varphi(t)\cos(At)] = \frac{1}{2}[F(u-A) + F(u+A)]. \tag{2.59}$$

$$\mathcal{F}[\varphi(t)\sin(At)] = \frac{1}{2i}[F(u-A) + F(u+A)]. \tag{2.60}$$

7. (Composition):

$$\int_{-\infty}^{\infty} F(u)h(u)e^{iut}dt = \int_{-\infty}^{\infty} \varphi(\xi)H(\xi - t)d\xi, \qquad H(u) = \mathcal{F}[h(t)]. \tag{2.61}$$

(Debnath, & Bhatta, 2015)

2.10 The relation between Sumudu and Fourier Transforms.

We can illustrate the relation between the Sumudu transform and the Fourier transform $\mathcal{F}(u)$ by the following theorem:

Theorem 2.2: (Khalaf & Belgacem, 2014)

Let $\varphi(t) \in B$, and let $S[\varphi(t)] = G(v)$, and $F(\varphi(t)] = \mathcal{F}(u)$, then,

$$\mathcal{F}(u) = \frac{1}{iu} G\left(\frac{1}{iu}\right). \tag{2.62}$$

$$G(v) = \frac{1}{iv} \mathcal{F}\left(\frac{1}{iv}\right). \tag{2.63}$$

Proof:

By equation (1.2)

$$\mathcal{F}(u) = F[\varphi(t)] = \int_{-\infty}^{\infty} e^{-iut}\varphi(t)dt \ , u \in R$$

Let $w = iut,\ t = \frac{w}{iu},\ \ dt = \frac{dw}{iu},$

$$\mathcal{F}(u) = F[\varphi(t)] = \int_{-\infty}^{\infty} e^{-w}\varphi\left(\frac{w}{iu}\right)\frac{dw}{iu},$$

$$= \frac{1}{iu}\int_{-\infty}^{\infty} e^{-w}\varphi\left(\frac{w}{iu}\right)dw = \frac{1}{iu}S[\varphi(w)](1/iu) = \frac{1}{iu}G(\frac{1}{iu})$$

$$\mathcal{F}(u) = \frac{1}{iu}G(\frac{1}{iu})$$

And by the Sumudu transform

$$G(v) = S[\varphi(t)] = \int_{0}^{\infty} \varphi(vt)e^{-t}dt,$$

Let $vt = z,\ t = \frac{z}{v}\ \ , dt = \frac{dz}{v},$ then

$$G(v) = S[\varphi(t)] = \int_{0}^{\infty} \varphi(z)e^{-iz/iv}\frac{dz}{v},$$

$$G(v) = \frac{1}{v}F[\varphi(z)](1/iv) = \frac{1}{v}\mathcal{F}\left(\frac{1}{iu}\right).$$

Then

$$G(v) = \frac{1}{v}\mathcal{F}\left(\frac{1}{iv}\right).$$

Example 2.4:(Debnath, & Bhatta, 2015)

We find the Sumudu transform and the Fourier transform of the Characteristic function $\mathrm{X}_{[-A,A]}(t) = H(A - |t|)$, where

$$\mathrm{X}_{[-A,A]}(t) = H(A - |t|) = \begin{cases} 1, & |t| < \mathrm{A}, \\ 0, & |t| > \mathrm{A}. \end{cases}$$

Where $H(t - A)$ is the Heaviside function.

By Sumudu transform:

$$G(v) = \frac{1}{v}\int_{0}^{\infty} e^{-t/v} dt = \frac{1}{v}\left[\frac{e^{-t/v}}{\frac{-1}{v}}\right]_{0}^{\infty} = -[0-1] = 1$$

By Fourier transform:

$$F_A(u) = \{\mathrm{H}_{[-A,A]}(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iut}\, \mathrm{X}_{[-A,A]}(t)dt$$

$$F(u) = \frac{1}{\sqrt{2\pi}}\int_{-A}^{\infty} e^{-iut}\, dt = \sqrt{\frac{2}{\pi}}\left(\frac{sinAu}{u}\right).$$

It can be noticed that when using the Sumudu transform, the result gives the integer value 1, which is equal to the value of the original function at $t < A$. But the Fourier transform give is the curve of the periodic function $(\sin\,(At))$.

2.11 Mellin Transform of Derivative.

1. $\mathcal{M}[\varphi'(x)] = (p-1)\tilde{\varphi}(p-1).$ (2.64)

2. $\mathcal{M}[\varphi''(x)] = (p-1)(p-2)\tilde{\varphi}(p-2).$ (2.65)

3. $\mathcal{M}[\varphi^n(t)] = (-1)^n \frac{\Gamma(p)}{\Gamma(p-n)}\, \mathcal{M}[\varphi(t), p\; - n].$ (2.66)

4. $\mathcal{M}[t^n\varphi^n(t)] = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\tilde{\varphi}(p).$ (2.67)

2.12 Some Properties of Mellin Transform.

1. Scaling property:

$$\mathcal{M}[a\varphi(t)] = a^{-p}\tilde{\varphi}(p), a > 0. \quad (2.68)$$

2. Shifting property:

$$\mathcal{M}[x^c\varphi(x)] = \tilde{\varphi}(p+c). \quad (2.69)$$

3. Convolution:

$$\mathcal{M}[\varphi(x) * g(x)] = \mathcal{M}\left[\int_0^\infty \varphi(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] = \tilde{\varphi}(p)\tilde{g}(p),$$

$$\mathcal{M}[\varphi(x)] = \tilde{\varphi}(p), \mathcal{M}[g(x)] = \tilde{g}(p). \quad (2.70)$$

4. Integrals property:

$$\mathcal{M}\left[\int_0^x \varphi(t)dt = \frac{-1}{p}\tilde{\varphi}\,(p-1).\right] \tag{2.71}$$

(Debnath, & Bhatta, 2015).

2.13 The Relation between Sumudu and Mellin Transforms.

To clarify the relationship between Sumudu and Mellin transforms, which is no less important than Sumudu and Fourier transforms. We first recall the relation between Laplace and Mellin transforms:

$$\mathcal{L}[\varphi(t)](s) = \mathcal{M}[\varphi(-\ln(x)](s). \tag{2.72}$$

$$\mathcal{M}[\varphi(x)](s) = \mathcal{L}[\varphi(e^{-t})](s). \tag{2.73}$$

The next theorem gives the relation between Sumudu and Mellin transforms.

Theorem 2.3: (Khalaf & Belgacem, 2014)

Let $\varphi(t) \in B$, then the relation between Sumudu transform $G(v)$ and Mellin transform $\mathcal{M}(s)$ is given by:

$$\mathcal{M}(s) = \mathcal{M}[\varphi(x)](s) = \mathcal{L}[\varphi(e^{-t})](s) = \frac{1}{s}S[\varphi(w)]\left(\frac{1}{s}\right) = \frac{G\left(\frac{1}{s}\right)}{s}. \tag{2.74}$$

Proof:

By the relation between Sumudu and Laplace transforms in equations (2.1) and (2.2), we have

$$S[\varphi(t)](v) = \frac{1}{v}\mathcal{L}[\varphi(w)]\frac{1}{v} \tag{2.75}$$

$$\mathcal{L}[\varphi(t)](s) = \frac{1}{s}S[\varphi(w)]\frac{1}{s} \tag{2.76}$$

By equations (2.72), (2.73), (2.75), and (2.76) we get

$$S[\varphi(t)](v) = \frac{1}{v}\mathcal{L}[\varphi(w)](1/v) = \frac{1}{v}\,\mathcal{M}[\varphi(-\ln(x)]\left(\frac{1}{v}\right). \tag{2.77}$$

Also

$$\mathcal{M}[\varphi(x)](s) = \mathcal{L}[\varphi(e^{-t})](s) = \frac{1}{s}S[\varphi(e^{-t})]\left(\frac{1}{s}\right). \tag{2.78}$$

(Debnath, & Bhatta, 2015)

Example 2.5: (Debnath, & Bhatta, 2015)

Let $\varphi(t) = t(e^t - 1)^{-1}$, then Sumudu transform of $\varphi(t)$ given as:

$$G(v) = S[t(e^t - 1)^{-1}] = \frac{1}{v}\int_0^\infty t(e^t - 1)^{-1} e^{-t/v} dt$$

By using

$$\sum_{m=0}^{\infty} e^{-mt} = \frac{1}{1 - e^t} ,$$

Then,

$$\sum_{m=1}^{\infty} e^{-mt} = \frac{-1}{e^t - 1},$$

$$G(v) = -\frac{1}{v}\sum_{m=1}^{\infty} \int_0^\infty te^{-t/v} e^{-mt} dt = -\frac{1}{v}\sum_{m=1}^{\infty}\int_0^\infty te^{-\left(\frac{1}{v}+m\right)t} dt,$$

Where

$$\int_o^\infty te^{-at} dt = \frac{1}{a^2}, a > 0$$

then

$$G(v) = \frac{1}{v}\sum_{m=1}^{\infty} \frac{1}{(m + 1/v)^2}.$$

Noting that

$$\sum_{m=1}^{\infty} \frac{1}{(m + a)^2} = \xi(2, a + 1).$$

Then

$$G(v) = \frac{1}{v}\xi\left(2, \frac{1}{v} + 1\right). \tag{2.79}$$

Now we notice the solution by using Mellin transform:

$$\mathcal{M}\left[\frac{t}{(e^t-1)}\right] = \tilde{\varphi}(p) = \int_0^\infty t^{p-1}\frac{t}{(e^t-1)}dt,$$

$$\mathcal{M}\left[\frac{t}{(e^t-1)}\right] = \sum_{m=1}^{\infty}\int_0^\infty t^p\, e^{-mt}\, dt,$$

$$= \sum_{m=1}^{\infty}\frac{\Gamma(p+1)}{m^{p+1}}$$

$$\mathcal{M}\left[\frac{t}{(e^t-1)}\right] = \Gamma(\mathrm{p}+1)\xi(\mathrm{p}+1). \tag{2.80}$$

Where

$$\xi(\mathrm{p}+1) = \sum_{m=1}^{\infty}\frac{1}{m^{p+1}}, (Re\, p > 1)$$

The famous Riemann Zeta function.

By solving using the two transforms, we note that the Sumudu transform is the best because it gave a clearer solution and is the easiest to deal with integrals and series.

Chapter Three

Fractional Sumudu Integral Transform.

3. Fractional Sumudu Integral Transform.

3.1 Introduction of the Fractional Calculus.

The Fractional calculus is an extension of the ordinary calculus, that its studies differential equations with non-integer orders. It came in 1695 from the scientists L' Hopital and Leibniz, who were at the beginning of questions about the fractional derivative and what the solution to differential equations that carry fractional orders. These questions have simulated other mathematicians, such as Riemann and others, to research this calculus. This science increased in development in the nineteenth century. When it appeared many theories of this calculation in most branches of engineering, modern sciences, chemical physics, statistics, and control theory, and it was considered that most of the problems in practical life are equations of the fractional degree.

Leibniz first introduced the nth derivative $\frac{d^n y}{dx^n} = D^n y$ where n is a non-negative integer, but if n is a fractional number, and then it is a paradox. The scientist Lacroix developed the formula for the nth derivative of $y = x^k$, in which m is a positive integer,

$$D^k y = \frac{l!}{(l-k)!} x^{l-k}, \qquad k \leq l$$

Replacing the factorial symbol with the gamma function:

$$D^\alpha x^\sigma = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + 1)} x^{\sigma - \alpha},$$

The above formula is the fractional derivative, where $\alpha \; and \; \sigma$ are fractional numbers.

The scientist Joseph Liouville (1832) presented an extended formula for the derivative of integral order n:

$$D^n e^{ax} = a^n e^{ax},$$

Replacing n by α, α is derivative of arbitrary order:

$$D^\alpha e^{ax} = a^\alpha e^{ax},$$

Liouville give this formula:

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x},$$

Where

$$f(x) = \sum_{n=0}^{\infty} c_n \exp(a_n x)\,, \quad Re\; a_n > 0.$$

This formula is known as Liouville first formula for the fractional derivative, and Liouville expanded his first definition and gave his second definition based on a gamma function. (Debnath, & Bhatta, 2015)

$$D^{\alpha} x^{-\sigma} = (-1)^{\alpha} \frac{\Gamma(\alpha + \sigma)}{\Gamma(\sigma)} x^{-\alpha-\sigma}, \quad \sigma > 0.$$

The second definition of the Liouville derivative is very useful for a class of rational functions. However, the first definition is very useful for a specific class of functions. Some scientists preferred the Lacroix formula, others preferred the Liouville dye, and many scientists have proposed different formulas for this calculation, including the Caputo formula, Modified Riemann-Liouville, Riesze operators, Hadmard, Grunwald, and Letnikov. However, the best and most popular definitions of fractional derivatives of order $\alpha\; > 0$ are Riemann-Liouville and Caputo. (AL-ZHOUR et al., 2019)

3.2 Riemann-Liouville Fractional Derivative:

Theorem 3.1: (Podlubny, 1999)

Let $\varphi(\tau)$ be a continuous function and integrable in the finite interval (b,t), and $\varphi(t)$ may be have integrable of order $r < 1$ at the point $\tau = b$,

$$\lim_{\tau \to b} (\tau - b)^r \, \varphi(t) = const(\neq 0)\,, \tag{3.1}$$

Then the integral

$$\varphi^{(-1)}(t) = \int_b^t \varphi(\tau) d\tau, \tag{3.2}$$

Exists and equals zero as $t \to b$.

Proof:

Let $\tau = b + z(t-b) \Longrightarrow d\tau = (t-b)dz,\; \varepsilon = t - b$

Then

$$\lim_{t \to b} \varphi^{-1}\,(t) = \lim_{t \to b} \int_b^t \varphi(\tau) d\tau$$

$$= \lim_{t \to b}(t-b)\int_0^1 \varphi(b+z(t-b)dz$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{1-r}\int_0^1 (\varepsilon z)^r\, \varphi(b+z\varepsilon)z^{-r}dz = 0$$

Because $r < 1$. Then the two-fold integral

$$\varphi^{(-2)}(t) = \int_b^t \varphi^{-1}(\tau)d\tau = \int_b^t \left(\int_b^\tau \varphi(\tau_1)d\tau_1\right)d\tau$$

$$= \int_b^t d\tau_1 \int_b^{\tau_1} \varphi(\tau)d\tau = \int_b^t \varphi(\tau)d\tau \int_\tau^t d\tau_1$$

$$= \int_b^t (t-\tau)\varphi(\tau)d\tau. \tag{3.3}$$

By integration (3.3), then the three-fold integral of $\varphi(\tau)$ is given by

$$\varphi^{(-3)}(t) = \int_b^t \varphi^{(-2)}(\tau)d\tau = \int_b^t \left(\int_b^\tau (t-\tau_1)\,\varphi(\tau_1)d\tau_1\right)d\tau$$

$$= \int_b^t d\tau_1 \int_{\tau_1}^t (\tau-\tau_1)\,\varphi(\tau_1)\,d(\tau) = \int_b^t \varphi(\tau_1)d\tau_1 \int_{\tau_1}^t (\tau-\tau_1)\,d(\tau),$$

$$= \int_b^t \frac{(t-\tau_1)^2}{2}\varphi(\tau_1)d\tau_1 = \frac{1}{2}\int_b^t (t-\tau)^2\,\varphi(\tau)d\tau.$$

By n terms, we can arrive at the Cauchy formula:

$$\varphi^{-n}(t) = \frac{1}{\Gamma(n)}\int_b^t (t-\tau)^{n-1}\varphi(\tau)\,d\tau. \tag{3.4}$$

Let $n \geq 1$, and take integer $m \geq 0$, then

$$\varphi^{(-m-n)}(t) = \frac{1}{\Gamma(n)}D^{-m}\int_b^t (t-\tau)^{n-1}\varphi(\tau)\,d\tau. \tag{3.5}$$

Where D^{-m} $(m \geq 0)$, denotes m iterated integrations.

$$\varphi^{(m-n)}(t) = \frac{1}{\Gamma(n)} D^m \int_b^t (t-\tau)^{n-1} \varphi(\tau)\, d\tau. \qquad (3.6)$$

Where D^m $(m \geq 0)$, denotes m iterated differentiations.

3.3 Derivative of Arbitrary Order.

In equation (3.6), it can replace integer n with a real α so that $m - \alpha > 0$, because the order $(m - n)$ in (3.6), the equation provides an opportunity for differentiation to non-integer order.

$$D_t^{m-\alpha} \varphi(t) = \frac{1}{\Gamma(\alpha)} \frac{d^m}{dt^m} \int_b^t (t-\tau)^{\alpha-1} \varphi(\tau) d\tau, \quad 0 < \alpha \leq 1, \qquad (3.7)$$

Let $q = m - \alpha \Longrightarrow \alpha = m - q$, then the **definition Riemann-Liouville derivative** is given by:

$$D_t^q \varphi(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_b^t (t-\tau)^{m-q-1} \varphi(\tau) d\tau, \qquad (m-1 < q \leq m), \qquad (3.8)$$

Or

$$D_t^q \varphi(t) = \frac{d^m}{dt^m} \left(D_t^{-(m-q)} \varphi(t) \right), \quad (m-1 \leq q < m). \qquad (3.9)$$

If $q = m - 1$, we obtain a conventional integer- order derivative of order $m - 1$

$$D_t^{m-1} \varphi(t) = \frac{d^m}{dt^m} \left(D_t^{-(m-(m-1)} \varphi(t) \right)$$

$$= \frac{d^m}{dt^m} (D_t^{-1} \varphi(t)) = \varphi^{(m-1)}(t).$$

When $q = m \geq 1$ and $t > b$

$$D_t^m \varphi(t) = \frac{d^m}{dt^m} (D_t^0 \varphi(t)) = \frac{d^m \varphi(t)}{dt^m} = \varphi^{(m)}(t). \qquad (3.10)$$

(Podlubny, 1999)

3.4 Some Properties of the Riemann-Liouville Derivative.

1. $D_t^q\left(D_t^{-q}\varphi(t)\right) = \varphi(t), \qquad q > 0, t > b$ (3.11)

2. $$D_t^{-q}\left(D_t^q\varphi(t)\right) = \varphi(t) - \sum_{i=1}^{m}\left[D_t^{q-i}\varphi(t)\right]\frac{(t-b)^{q-i}}{\Gamma(q-i+1)}, \quad t = b. \tag{3.12}$$

3. Linearity:

$$D_t^q\left(\lambda f(t) + \mu g(t)\right) = \lambda D_t^q f(t) + \mu D_t^q g(t). \tag{3.13}$$

Proof:

1. Let $q = n \geq 1$, then

$$D_t^n\left(D_t^{-n}\varphi(t)\right) = \frac{d^n}{dt^n}\int_b^t (t-\tau)^{n-1}\,\varphi(\tau)d\tau$$

$$= \frac{d}{dt}\int_b^t \varphi(\tau)d\tau = \varphi(t)$$

Now let $m - 1 < q < m$, and using the composition rule of the **Riemann-Liouville fractional integral**:

$$D_t^{-q}\left(D_t^{-p}\varphi(t)\right) = D_t^{-p}\left(D_t^{-q}\varphi(t)\right) = D_t^{-q-p}\varphi(t), \tag{3.14}$$

Then it can write

$$D_t^{-m}\varphi(t) = D_t^{-(m-q)}(D_t^{-q}\varphi(t))$$

Then

$$D_t^q\left(D_t^{-q}\varphi(t)\right) = \frac{d^m}{dt^m}\left\{D_t^{-(m-q)}(D_t^{-q}\varphi(t))\right\}$$

$$= \frac{d^m}{dt^m}\{D_t^{-m}\varphi(t)\} = \varphi(t).$$

2. $$D_t^{-q}\left(D_t^q\varphi(t)\right) = \frac{1}{\Gamma(q)}\int_b^t (t-\tau)^{q-1}D_\tau^q\varphi(\tau)\,d\tau.$$

$$= \frac{d}{dt}\left\{\frac{1}{\Gamma(q+1)}\int_b^t (t-\tau)^q D_\tau^q\varphi(\tau)\,d\tau\right\}. \tag{3.15}$$

Integration by parts, and using equation (3.14):

$$\frac{1}{\Gamma(q+1)}\int_b^t (t-\tau)^q D_\tau^q \varphi(\tau)\, d\tau = \frac{1}{\Gamma(q+1)}\int_b^t (t-\tau)^q \frac{d^m}{d\tau^m}\left\{ D_\tau^{-(m-q)} \varphi(\tau)\right\} d\tau$$

$$= \frac{1}{\Gamma(q-m+1)}\int_b^t (t-\tau)^{q-m} \frac{d^m}{d\tau^m}\left\{ D_\tau^{-(m-q)} \varphi(\tau)\right\} d\tau$$

$$-\sum_{i=1}^{m}\left[\frac{d^{m-i}}{dt^{m-i}}\left(D_\tau^{-(m-q)} \varphi(t)\right)\right]\frac{(t-b)^{q-i+1}}{\Gamma(2+q-i)}$$

$$= \frac{1}{\Gamma(q-m+1)}\int_b^t (t-\tau)^{q-m}\left\{ D_\tau^{-(m-q)} \varphi(\tau)\right\} d\tau$$

$$-\sum_{i=1}^{m}\left[\left(D_\tau^{q-i} \varphi(t)\right)\right]\frac{(t-b)^{q-i+1}}{\Gamma(2+q-i)}, \tag{3.16}$$

$$= D_\tau^{-(q-m+1)}\left(D_\tau^{-(m-q)} \varphi(t)\right)$$

$$-\sum_{i=1}^{m}\left[\left(D_\tau^{q-i} \varphi(t)\right)\right]\frac{(t-b)^{q-i+1}}{\Gamma(2+q-i)}, \tag{3.17}$$

$$= D_\tau^{-1}\varphi(t) - \sum_{i=1}^{m}\left[\left(D_\tau^{q-i} \varphi(t)\right)\right]\frac{(t-b)^{q-i+1}}{\Gamma(2+q-i)}, \tag{3.18}$$

Combining (3.15) and (3.18) to end the proof of the property. When $0 < q < 1$ then:

$$D_t^{-q}\left(D_t^q \varphi(t)\right) = \varphi(t) - \left[D_t^{q-1}\varphi(t)\right]\frac{(t-b)^{q-1}}{\Gamma(q)}. \tag{3.19}$$

It can be considered that the equation (3.11) is a particular case of a general property:

$$D_t^q\left(D_t^{-p} f(t)\right) = D_t^{q-p}\varphi(t), \tag{3.20}$$

Two case let: $q \geq p \geq 0$, and using (3.14) and (3.11),

$$D_t^q\left(D_t^{-p}\varphi(t)\right) = D_t^q\left(D_t^{-p} D_t^{-(p-q)}\varphi(t)\right)$$

$$= D_t^{-(p-q)} = D_t^{q-p}\varphi(t).$$

And case $q > p \geq 0$. let us denote by m and n integers, where $0 \leq k-1 \leq q < k$ and $0 \leq n \leq q-p < n,\ n \leq m$, and using (3.14) and (3.8),

$$D_t^q\left(D_t^{-p}f(t)\right)=\frac{d^k}{dt^k}\left\{D_t^{-(k-q)}\left(D_t^{-p}\varphi(t)\right)\right\}$$
$$=\frac{d^k}{dt^k}\left\{D_t^{q-p-k}\varphi(t)\right\}$$
$$=\frac{d^k}{dt^k}\left\{D_t^{q-p-k}\varphi(t)\right\}=D_t^{q-p}\varphi(t).$$

It can consider equation (3.12) particular case of the following,

$$D_t^q\left(D_t^{-p}\varphi(t)\right)=D_t^{p-q}\varphi(t)-\sum_{i=1}^{m}\left[D_t^{p-i}\varphi(t)\right]\frac{(t-b)^{q-i}}{\Gamma(1+q-i)},\qquad(3.21)$$

3. For $\lambda,\mu\in\mathbb{R}$, we have

$$D_t^q\big(\lambda f(t)+\mu g(t)\big)=\frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_b^t(t-\tau)^{m-q-1}(\lambda f(\tau)+\mu g(t))d\tau$$

$$=\frac{\lambda}{(\Gamma(m-q)}\frac{d^m}{dt^m}\int_b^t(t-\tau)^{m-q-1}f(\tau)d\tau+\frac{\mu}{(\Gamma(m-q)}\frac{d^m}{dt^m}\int_b^t(t-\tau)^{m-q-1}g(\tau)d\tau$$

$=\lambda D_t^q f(t)+\mu D_t^q g(t)$. (Podlubny, 1999)

3.5 Caputo's Fractional Derivative.

Most applied problems in physics, especially in solid mechanics and the theory of viscosity, require definitions of the fractional derivative through initial conditions that allow for physical interpretation and contain $f(x), f'(x)\ldots$

Although such problems are solved mathematically successfully using the Riemann-Liouville approach, some of their solutions are not practically logical, i.e., it has no clear physical explanation.

Caputo's approach came to suggest a solution to this problem, and two years later, he presented his famous definition that states the following:

Definition 3.1: (Podlubny, 1999)

Let $\varphi(t)$ be a continuous function on a closed interval $[0,t]$ and $\alpha\in(m-1,m), m\in N$, then the definition of Caputo derivative is given by:

$${}_a^cD_t^\alpha\varphi(t)=\frac{1}{\Gamma(\alpha-m)}\int_b^t\frac{\varphi^m(t)d\tau}{(t-\tau)^{\alpha-m+1}},\qquad m-1<\alpha<m.\qquad(3.22)$$

3.6 Some Properties of Caputo Fractional Derivative.

1. ${}_0^cD_t^0\varphi(t) = \varphi(t).$ (3.23)

2. ${}_0^cD_t^\alpha\varphi(t) = \varphi^n(t), \quad n \in N$. (3.24)

3. ${}_0^cD_t^\alpha\big(\lambda f(t) + \mu g(t)\big) = \lambda{}_0^cD_t^\alpha f(t) + \mu{}_0^cD_t^\alpha g(t).$ (3.25)

Proof:

1. If $\alpha = 0$, then $m = 1$, by using (3.22):

$${}_0^cD_t^0\varphi(t) = \frac{1}{\Gamma(1)}\int_0^t \frac{\varphi(\tau)d\tau}{(t-\tau)^{-1+1}} = \int_0^t \varphi'(\tau)d\tau = \varphi(t-0) = \varphi(t)$$

2. If $\alpha \in (m-1, m)$, it's follows that $\alpha = m-1 \Rightarrow n = m-1$, then

$${}_0^cD_t^\alpha\varphi(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{\varphi^m(\tau)d\tau}{(t-\tau)^{\alpha-m+1}} = \frac{1}{\Gamma(m-m+1)}\int_0^t \frac{\varphi^m(\tau)d\tau}{(t-\tau)^{m-1-m+1}} =$$

$$= \frac{1}{\Gamma(1)}\int_0^t \varphi^m(\tau)\, d\tau = \varphi^{m-1}(t) = \varphi^n(t).$$

Prove of the linear property (3) is similar to prove of the linear property using the Riemann-Liouville derivative. (Sikora, 2023)

3.7 Special Functions of the Fractional Calculus.

Before starting in study fractional Sumudu transform, some special functions used in fractional calculus will be presented, which play an important role in simplifying problems and arriving at a final solution to fractional differential equations, the most important of which are the Gamma function, the Beta function, and the Mittag-Leffler function.

3.7.1 Gamma function.

The Gamma function $\Gamma(m)$ is defined by:

$$\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}dt. \quad (3.26)$$

One of the important basic properties of the gamma function is given:

$$\Gamma(m+1) = m\Gamma(m) = m!, \quad (3.27)$$

It can prove this by integration by parts:

$$\Gamma(m+1) = \int_0^\infty t^m e^{-t}\, dt = [-e^{-t}t^2]_0^\infty + m\int_0^\infty t^{m-1}e^{-t}dt = m\Gamma(m)$$

Then

$$\Gamma(m+1) = m.(m-1) = m!$$

3.7.2 Beta function.

The beta function $\beta(k,n)$ is defined by:

$$\beta(k,n) = \int_0^1 (1-\tau)^{k-1}\tau^{n-1}d\tau, \quad (Re(k) > 0, \quad Re(n) > 0) \tag{3.28}$$

To drive the relationship between the Beta and Gamma functions, we consider the following integer:

$$h_{k,n}(t) = \int_0^1 (1-\tau)^{k-1}\tau^{n-1}d\tau,$$

Which is a convolution of the function τ^{k-1} and $(1-\tau)^{n-1}$, and using the Laplace transform of a convolution, we obtain:

$$H_{k,n}(s) = \frac{\Gamma(k)}{s^k}\,.\frac{\Gamma(n)}{s^n} = \frac{\Gamma(k)\Gamma(n)}{s^{k+n}}.$$

Taking the invers Laplace transform yields:

$$h_{k,n}(t) = \frac{\Gamma(k)\Gamma(n)}{\Gamma(k+n)}t^{k+n-1},$$

Setting $t = 1$, we get

$$\beta(k,n) = \frac{\Gamma(k)\Gamma(n)}{\Gamma(k+n)}\,. \tag{3.29}$$

3.7.3 Mittag-Leffler function.

The Mittag-Leffler function is one of the important functions in fractional calculus and is defined as:

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{n}{\Gamma(n\alpha+1)}, \qquad \alpha > 0 \tag{3.30}$$

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha, \beta > 0 \tag{3.31}$$

Her, we will present some properties of Mittag-Leffler function:

$$\mathbf{1.}\, E_{1,1}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} = e^t. \tag{3.32}$$

$$\mathbf{2.}\, E_{1,2}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+2)} = \sum_{n=0}^{\infty} \frac{t^n}{(n+2)!} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)} = \frac{e^t - 1}{t}. \tag{3.33}$$

$$\mathbf{3.}\, E_{1,m}(t) = \frac{1}{t^{m-1}} \left\{ e^t - \sum_{n=0}^{m-2} \frac{t^n}{n!} \right\}. \tag{3.34}$$

$$\mathbf{4.}\, E_{2,1}(t^2) = \sum_{n=0}^{\infty} \frac{t^{2n}}{\Gamma(2n+1)} = \sum_{n=0}^{\infty} \frac{e^{2n}}{(2n)!} = \cosh(t). \tag{3.35}$$

$$\mathbf{5.}\, E_{2,2}(t^2) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(2n+2)} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = \frac{\sinh(t)}{t}. \tag{3.36}$$

$$\mathbf{6.}\, E_{1/2,1}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma\left(\frac{n}{2} + 1\right)} = e^{t^2} erf(-t). \tag{3.37}$$

Where $erf(t)$ is the error function and is defined by:

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-z^2}\, dt, \tag{3.38}$$

$$erf(t) = \frac{2}{\sqrt{\pi}} e^{-t^2} \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!} t^{2n+1}. \tag{3.39}$$

(Podlubny, 1999)

Example 3.1: (Sikora, 2023)

Calculate the fractional derivative ${}_0^cD^{\frac{1}{2}}t$ by using Caputo's definition.

Solution:

$${}_0^cD_t^{\frac{1}{2}} t = \frac{1}{\Gamma(1 - \frac{1}{2})} \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}}}\, d\tau,$$

Integration by parts:

$$u = t - \tau, du = -d\tau \Rightarrow d\tau = -du$$

In addition, the bounds of the integral:

$$\tau \in (0, t) \Rightarrow u \in (t, 0)$$

$${}_0^c D_t^{\frac{1}{2}} t = \frac{1}{\sqrt{\pi}} \int_t^0 \frac{-1}{(u)^{\frac{1}{2}}} du = \frac{1}{\sqrt{\pi}} \int_0^t (u)^{-1/2} du = \frac{1}{\sqrt{\pi}} \left[2\sqrt{u}\right]_0^t = \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

Corollary 3.1:(Sikora, 2023)

The following formula defines the Caputo fractional derivative of the power function:

$${}_0^c D_t^{\alpha} t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, & p > m-1, p \in R \\ 0, & p \leq m-1, p \in N \end{cases} \tag{3.40}$$

Proof: The solution it starts with $p > k - 1$

$${}_a^c D_t^{\alpha} t^p = \frac{1}{\Gamma(m-\alpha)} \int_c^t \frac{\varphi^m(t) d\tau}{(t-\tau)^{\alpha-m+1}}$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\Gamma(p+1)}{\Gamma(p-m+1)} \tau^{p-m} (t-\tau)^{m-\alpha-1} d\tau.$$

Let $\tau = \lambda t$, $\lambda = \frac{\tau}{t}$, for $0 \leq \lambda \leq 1$, then

$${}_a^c D_t^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(m-\alpha)(p-m+1)} \int_0^t (\lambda t)^{p-m} ((1-\lambda)t)^{m-\alpha-1} d\lambda.$$

$$= \frac{\Gamma(p+1)}{\Gamma(m-\alpha)(p-m+1)} t^{p-\alpha} \int_0^t (\lambda)^{p-m} (1-\lambda)^{m-\alpha-1} d\lambda.$$

$$= \frac{\Gamma(p+1)}{\Gamma(m-\alpha)(p-m+1)} t^{p-\alpha} \beta(p-m+1, m-\alpha)$$

$$= \frac{\Gamma(p+1)}{\Gamma(m-\alpha)(p-n+1)} t^{p-\alpha} \frac{\Gamma(p-m+1)\Gamma(m-\alpha)}{\Gamma(p-\alpha+1)}$$

$$= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}.$$

For $p \leq m - 1, (t^p)^{(m)} = 0$, hence $D_t^{\alpha} t^p = 0$

Example 3.2:(Sikora, 2023)

Calculate the Caputo fractional derivative of the function $\varphi(t) = t^3$,

where $\alpha = \frac{2}{3}$, $p = 3$

Solution:

By using equation (3.40)

$$ {}_0^cD_t^{\frac{2}{3}} t^3 = \frac{\Gamma(4)}{\Gamma(\frac{10}{3})} t^{7/3} = \frac{3!}{\Gamma(\frac{10}{3})} t^{7/3} = \frac{6}{\Gamma(\frac{10}{3})} t^{7/3}. $$

Theorem 3.2: (Sikora, 2023)

The exponential function by definition Caputo derivative is given by:

$$ {}_a^cD_t^{\alpha} e^{at} = \sum_{n=0}^{\infty} \frac{a^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = a^n t^{n-\alpha} E_{1,n-\alpha+1}(at), \qquad (3.41) $$

Where $E_{\alpha,\beta}$ is the Mittag-Leffler function of two parameters (3.31)

Proof: It can prove that by definition Riemann-Liouville:

Example 3.3:(Sikora, 2023)

Find the fractional Caputo's derivative of the function $\varphi(t) = e^t$ of order $\alpha = \frac{1}{2}$.

Solution:

By equation (3.41), $a = 1$,

$$ {}_0^cD_t^{1/2} = t^{1/2} E_{1,\frac{3}{2}}(t) = \sqrt{t} E_{1,\frac{3}{2}}(t). $$

$$ E_{1,\frac{3}{2}}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma\left(n + \frac{3}{2}\right)} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{t^{n+1}}{\Gamma\left(n + 1 + \frac{1}{2}\right)} = \frac{1}{t} \sum_{m=0}^{\infty} \frac{t^{n+1}}{\frac{\sqrt{\pi}\,(2n+1)!}{2^{n+1}}} $$

$$ = \frac{1}{t\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{t^{n+1} 2^{n+1}}{(2n+1)!} = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{t^n 2^n}{(2n+1)!} $$

$$ = \frac{e^t erf\,(\sqrt{t})}{\sqrt{t}}. $$

Then

$$ {}_0^cD_t^{1/2} = \sqrt{t}E_{1,\frac{3}{2}}(t) = \sqrt{t}\frac{e^t erf\,(\sqrt{t})}{\sqrt{t}} = e^t\, erf\big(\sqrt{t}\big). $$

3.8 Fractional Sumudu Integral Transform.

Introduction:

As noted earlier, the fractional Sumudu transform its importance in both scientific and practical fields. The fractional Sumudu transform is considered one of the integral transforms that provides greater flexibility in solving fractional differential equations and analyzing signals. This advantage has led to its widespread use in the fields of control engineering, physics, and applied mathematics.

In this chapter, we will study the fractional Sumudu transform and introduce some of its properties, and we will introduce some fundamental theorems.

Definition 3.2:(AL-Zhour et al., 2019)

The fractional Sumudu transform is defined over the set:

$$ B_\alpha = \{\varphi(t)|\exists M, \tau_1, \tau_2 > 0. |\varphi(t)| < Me^{\frac{|t^\alpha|}{\alpha\tau_i}}, if\ t^\alpha \in (-1)^i \times [0,\infty), i = 1,2\} $$

By

$$ G(v) = S_\alpha[\varphi(t)] = \frac{1}{v}\int_0^\infty e^{\frac{-t^\alpha}{\alpha v}}\ \varphi(t)d_\alpha(t). \tag{3.42} $$

Where $d_\alpha(t) = t^{\alpha-1}$ $0 < \alpha < 1$.

Theorem 3.3:(AL-Zhour et al., 2019)

Let $\varphi(t) \in B_\alpha, \varphi: [0.\infty) \to R$, then the fractional Sumudu transform of derivative functions is given by:

$$ S_\alpha[D^{n\alpha}\varphi(t)] = \frac{G_\alpha(v)}{v^n} - \frac{\varphi(0)}{v^n}, \quad 0 < \alpha < 1, \qquad n \in N. \tag{3.43} $$

Proof:

We start by proving the first derivative by using the definition of the Laplace transform, as

$$ S_\alpha[D^\alpha \varphi(t)] = \frac{\mathcal{L}_\alpha[D^\alpha\varphi(t)]_{s\to\frac{1}{v}}}{v} = \frac{[sF(s) - \varphi(0)]_{s\to\frac{1}{v}}}{v}, $$

$$ = \frac{\frac{1}{v}G_\alpha\left(\frac{1}{v}\right)}{v} - \frac{\varphi(0)}{v} = \frac{G_\alpha(v)}{v} - \frac{\varphi(0)}{v}. $$

Similarly, we can prove:

$$S_\alpha[D^{n\alpha}\varphi(t)] = \frac{G_\alpha(v)}{v^n} - \frac{\varphi(0)}{v^n},$$

3.9 Some Properties of the Fractional Sumudu Transform.

Suppose that $a,,\lambda,\mu \in R$ and $0 < \alpha \le 1$. Then:

$$1. S_\alpha[a] = a. \tag{3.44}$$

$$2. S_\alpha\left[e^{a\frac{t^\alpha}{\alpha}}\right] = \frac{1}{1-\alpha v}, v > \frac{1}{a}. \tag{3.45}$$

$$3. S_\alpha\left[\sin\left(a\frac{t^\alpha}{\alpha}\right)\right] = \frac{av}{1+a^2v^2}, v > \frac{1}{|a|}. \tag{3.46}$$

$$4. S_\alpha\left[\cos\left(a\frac{t^\alpha}{\alpha}\right)\right] = \frac{1}{1+a^2v^2}, v > \frac{1}{|a|}. \tag{3.47}$$

$$5. S_\alpha\left[\sinh\left(a\frac{t^\alpha}{\alpha}\right)\right] = \frac{av}{1-a^2v^2}, v > \frac{1}{|a|}. \tag{3.48}$$

$$6. S_\alpha\left[\cosh\left(a\frac{t^\alpha}{\alpha}\right)\right] = \frac{1}{1-a^2v^2}, v > \frac{1}{|a|}. \tag{3.49}$$

$$7. S_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\right] = \Gamma(n+1)v^n, v > 0. \tag{3.50}$$

8. **Linear property:**

$$S_\alpha\{\lambda f(t) + \mu g(t)\} = \lambda F_\alpha(v) + \mu G_\alpha(v), \tag{3.51}$$

9. **Shifting property:**

$$S_\alpha\left[e^{-a\frac{t^\alpha}{\alpha}}\varphi(t)\right] = \frac{F_\alpha\left(\frac{1}{v}+a\right)}{v}. \tag{3.52}$$

10. **Integral property:**

$$S_\alpha = [I^\alpha\varphi(t)] = F_\alpha\left(\frac{1}{v}\right). \tag{3.53}$$

11. **Convolution property:**

$$S_\alpha(f * g) = vF_\alpha(v)G_\alpha(v). \tag{3.54}$$

12. **Power product property:**

$$S_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\varphi(t)\right] = \frac{1}{v}\left[(-1)^n\frac{d^n}{ds^n}F_\alpha(s)\right], s \to \frac{1}{v} \tag{3.55}$$

Proof:

$$\mathbf{1}. S_\alpha[a] = \frac{\mathcal{L}_\alpha[a]_{s\to\frac{1}{v}}}{v} = \frac{\left[\frac{a}{s}\right]_{s\to\frac{1}{v}}}{v} = a$$

$$\mathbf{2}.\, S_\alpha\left[e^{a\frac{t^\alpha}{\alpha}}\right] = \frac{\mathcal{L}_\alpha\left[e^{a\frac{t^\alpha}{\alpha}}\right]_{s\to\frac{1}{v}}}{v} = \frac{\mathcal{L}_\alpha[e^{at}]_{s\to\frac{1}{v}}}{v} = \frac{\left[\frac{1}{s-a}\right]_{s\to\frac{1}{v}}}{v} = \frac{1}{1-av}.$$

$$\mathbf{3}.\, S_\alpha\left[\sin\left(a\frac{t^\alpha}{\alpha}\right)\right] = \frac{\mathcal{L}_\alpha\left[\sin\left(a\frac{t^\alpha}{\alpha}\right)\right]_{s\to\frac{1}{v}}}{v} = \frac{\mathcal{L}_\alpha[\sin(at)]_{s\to\frac{1}{v}}}{v}$$

$$= \frac{\left[\frac{a}{s^2+a^2}\right]_{s\to\frac{1}{v}}}{v} = \frac{av}{1+a^2v^2}.$$

Similarly, by the same steps in property (3), we can prove the properties (4), (5), and (6)

$$\mathbf{7}.\, S_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\right] = \frac{\mathcal{L}_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\right]_{s\to\frac{1}{v}}}{v} = \frac{\Gamma(n+1)v^{n+1}}{v} = \Gamma(n+1)v^n.$$

8. We can prove the linear property of the fractional Sumudu transform by the same steps as the linear property of the Sumudu transform by using the definition or the relation.

$$\mathbf{9}.\, S_\alpha\left[e^{-a\frac{t^\alpha}{\alpha}}\varphi(t)\right] = \frac{\mathcal{L}_\alpha\left[e^{-a\frac{t^\alpha}{\alpha}}\varphi(t)\right]_{s\to\frac{1}{v}}}{v} = \frac{\mathcal{L}_\alpha\left[e^{-at}\varphi(\alpha t)^{\frac{1}{\alpha}}\right]_{s\to\frac{1}{v}}}{v} = \frac{F_\alpha\left(\frac{1}{v}+a\right)}{v}.$$

$$\mathbf{10}.\, S_\alpha[I^\alpha\varphi(t)] = \frac{\mathcal{L}_\alpha[I^\alpha\varphi(t)]_{s\to\frac{1}{v}}}{v} = vF_\alpha(v)$$

$$\mathbf{11}.\, S_\alpha(f*g) = \frac{\mathcal{L}_\alpha[(f*g)]_{s\to\frac{1}{v}}}{v} = \frac{[F_\alpha(s)*G_\alpha(s)]_{s\to\frac{1}{v}}}{v} = \frac{F_\alpha\left(\frac{1}{v}\right)G_\alpha\left(\frac{1}{v}\right)}{v}$$
$$= vF_\alpha(v)G_\alpha(v)$$

$$\mathbf{12}.\, S_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\varphi(t)\right] = \frac{\mathcal{L}_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\varphi(t)\right]_{s\to\frac{1}{v}}}{v} = \frac{1}{v}\mathcal{L}_\alpha\left[t^n\varphi(\alpha t)^{\frac{1}{\alpha}}\right]_{s\to\frac{1}{v}}$$
$$= \frac{1}{v}\left[(-1)^n\frac{d^n}{ds^n}\,F_\alpha(s)\right]_{s\to\frac{1}{v}}.$$

(AL-ZHOUR et al., 2019)

3.10 Sumudu Transform of the Riemann-Liouville Fractional Derivative.

Theorem 3.4:(Bodkhe & Panchal, 2016)

Let $\varphi(t) \in B$, the Sumudu transform of $\varphi(t)$ is G, and let $n \in N, n-1 \leq \alpha < n$, then the **Sumudu transform of the Riemann-Liouville fractional derivative** of the function is given by:

$$S[D_t^{\alpha}\varphi(t)] = v^{-\alpha}G(v) - \sum_{m=0}^{n-1} v^{-(m+1)}[D_t^{\alpha-m-1}\varphi(t)]_{t=0}\,, \tag{3.56}$$

Proof: Suppose that $D_t^{\alpha}\varphi(t) = h^n(t) = \frac{d^n}{dt^n}h(t)$ then.

$$h(t) = \frac{d^{-n}}{dt^{-n}}\frac{d^n}{dt^n}h(t)$$

$$= \frac{d^{-n}}{dt^{-n}}D_t^{\alpha}\varphi(t)$$

$$h(t) = D_t^{-(n-\alpha)}\varphi(t)$$

Using the Sumudu transform

$$G_{\alpha}(v) = S[h(t)] = S\left[D_t^{-(n-\alpha)}\varphi(t)\right](v) = v^{(n-\alpha)}G(v). \tag{3.57}$$

And by equation (1.14)

$$S[D_t^{\alpha}\varphi(t)](v) = S\left[\frac{d^n}{dt^n}h(t)\right]$$

$$= \frac{G_{\alpha}(v)}{v^n} - \sum_{m=0}^{n-1}\frac{h^{(n-m-1)}(t)|t=0}{v^{m+1}}, \tag{3.58}$$

By the definition of the Riemann-Liouville fractional derivative (3.14), we have

$$h^{(n-m-1)}(t) = \frac{d^{n-m-1}}{dt^{n-m-1}}h(t)|_{t=0}$$

$$= \frac{d^{n-m-1}}{dt^{n-m-1}}\left[D_t^{-(n-\alpha)}\varphi(t)\right]_{\boldsymbol{t=0}}$$

$$= D_t^{n-m-1}\left[D_t^{-(n-\alpha)}\varphi(t)\right]_{\boldsymbol{t=0}}$$

$$= D_t^{\alpha-m-1}\varphi(t)|_{t=0} \tag{3.59}$$

Then, by using equations (3.57) and (3.59) in equation (3.58)

$$S[D_t^{\alpha}\varphi(t)] = G_{\alpha}(v) = v^{-\alpha}G(v) - \sum_{m=0}^{n-1} v^{-(m+1)}[D_t^{\alpha-m-1}\varphi(t)]_{t=0}$$

3.11 Sumudu Transform of the Caputo Fractional Derivative.

Theorem 3.5: (Bodkhe & Panchal, 2016)

Let $\varphi(t) \in B$, the Sumudu transform of $\varphi(t)$ is G, and let $k \in N, k-1 < \alpha < k$, then the Sumudu transform of **the Caputo fractional derivative** of the function $\varphi(t)$ is given by:

$$S[{}_0^cD_t^{\alpha}\varphi(t)] = G_{\alpha}^c(v) = v^{-\alpha}\left[G(v) - \sum_{m=0}^{n-1} v^n[\varphi^n(t)]_{t=0}\right], -1 < k-1 \leq \alpha < k. \tag{3.60}$$

Proof:

Let $g(t) = \varphi^m(t)$ and, by definition of the Caputo derivative,

$$\begin{aligned} {}_0^cD_t^{\alpha}\varphi(t) &= \frac{1}{\Gamma(k-\alpha)}\int_0^t \frac{\varphi^k(\tau)d\tau}{(t-\tau)^{\alpha-k+1}} \\ &= \frac{1}{\Gamma(k-\alpha)}\int_0^t \frac{g(\tau)d\tau}{(t-\tau)^{\alpha-k+1}} \\ &= D_t^{-(k-\alpha)}g(t) \end{aligned} \tag{3.61}$$

Applying the Sumudu transform to equation(3.61), one has

$$S[{}_0^cD_t^{\alpha}\varphi(t)] = S\left[D_t^{-(k-\alpha)}g(t)\right](v) = v^{(-\alpha)}G(v)$$

$$S[g(t)] = S[\varphi^k(t)]$$

Hence, equation (1.14) becomes

$$G(v) = \frac{G(v)}{v^k} - \sum_{n=0}^{k-1}\left[\frac{\varphi^n(t)}{v^{k-n}}\right]_{t=0}$$

$$S[{}_{0}^{c}D_{t}^{\alpha}\varphi(t)] = v^{k-\alpha}\left\{\frac{G(v)}{v^{k}} - \sum_{n=0}^{k-1}\left[\frac{f^{n}(t)}{v^{k-n}}|_{t=0}\right]\right\},$$

$$= v^{-\alpha}G(v) - \sum_{n=0}^{k-1} v^{n-\alpha}\,[\varphi^{n}(t)]_{t=0}$$

Then

$$G_{\alpha}^{c}(v) = v^{-\alpha}\left[G(v) - \sum_{n=0}^{k-1} v^{n}[\varphi^{n}(t)]_{t=0}\right], -1 < k-1 \leq \alpha < k.$$

Collaraly 3.2: (Hamza & Elzaki, 2015)

We can obtain the Sumudu transform by Caputo fractional derivative for $m-1 < \alpha \leq m, m \in N$, by

$$S[D^{\alpha}\varphi(t)] = v^{m-\alpha}\left[\frac{G(v)}{v^{m}} - \frac{\varphi(0)}{v^{m}} - \frac{\varphi'(0)}{v^{m-1}} - \cdots - \frac{\varphi^{(m-1)}}{v}\right], \tag{3.62}$$

Collaraly 3.3: (Hamza & Elzaki, 2015)

For $a \in R$, $\alpha, l > 0$, $\frac{1}{v^{\alpha}} > |a|$, then it has the inverse Sumudu transform of next formula:

$$S^{-1}\left[\frac{v^{l-1}}{1+av^{\alpha}}\right] = t^{l-1}E_{\alpha,l}(-at^{\alpha}). \tag{3.63}$$

Proof: By applying the series expansion, then

$$\frac{v^{l}}{1+av^{\alpha}} = v^{l}\frac{1}{1+av^{\alpha}} = v^{l}\sum_{n=0}^{\infty}(-av^{\alpha})^{n},$$

Then

$$\frac{v^{l-1}}{1+av^{\alpha}} = \sum_{n=0}^{\infty}(-a)^{n}v^{n\alpha+l-1}$$

Taking inverse the Sumudu transform:

$$\sum_{n=0}^{\infty}\frac{(-a)^{n}v^{n\alpha+l-1}}{\Gamma(n\alpha+l)} = v^{l-1}\sum_{n=0}^{\infty}\frac{(-at^{\alpha})^{n}}{\Gamma(n\alpha+l)}$$

$$= t^{l-1}E_{\alpha,l}(-at^{\alpha}).$$

Collaraly 3.4:(Hamza & Elzaki, 2015)

For $\alpha, \beta > 0, \alpha \geq \beta > 0$, and $a \in R$, then

$$S^{-1}\left[\frac{v^{\alpha(k+1)-1}}{(1+av^{\alpha-\beta})^{k+1}}\right] = t^{\alpha(k+1)-1}\sum_{n=0}^{\infty}\frac{(-a)^n\binom{k+n}{n}}{\Gamma(n(\alpha-\beta)+(k+1)\alpha)}t^{n(\alpha-\beta)}.$$

(3.64)

Proof:

Using the series expansion of $(1+t)^{-k-1}$, then

$$\frac{1}{(1+t)^{k+1}} = \sum_{n=0}^{\infty}\binom{k+n}{n}(-t)^n$$

We have

$$\frac{v^{\alpha(k+1)-1}}{(1+av^{\alpha-\beta})^{k+1}} = v^{\alpha(k+1)-1}\frac{1}{(1+av^{\alpha-\beta})^{k+1}} = v^{\alpha(k+1)-1}\sum_{n=0}^{\infty}\binom{k+n}{n}(-av^{\alpha-\beta})^n$$

Taking the inverse Sumudu transform,

$$S^{-1}\left[v^{\alpha(k+1)-1}\sum_{n=0}^{\infty}\binom{k+n}{n}(-av^{\alpha-\beta})^n\right]$$
$$= t^{\alpha(k+1)-1}\sum_{n=0}^{\infty}\frac{(-a)^n\binom{k+n}{n}}{\Gamma(n(\alpha-\beta)+(k+1)\alpha)}t^{n(\alpha-\beta)}.$$

Collaraly 3.5:(Hamza & Elzaki, 2015)

For $\alpha \geq \beta, \alpha > \sigma, a \in R$ and $\left|\frac{v^{\alpha-1}}{1+av^{\alpha-\beta}}\right| > |b|$, we obtain

$$S^{-1}\left[\frac{v^{\alpha+\beta-\sigma-1}}{v^{\beta}+av^{\alpha}+bv^{\alpha+\beta}}\right] = t^{\alpha-\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\sigma)}t^{k(\alpha-\beta)+n\alpha}$$

(3.65)

Proof: By using series expansion can be written as:

$$\frac{v^{\alpha+\beta-\sigma-1}}{v^{\beta}+av^{\alpha}+bv^{\alpha+\beta}}=\frac{v^{\alpha+\beta-\sigma-1}}{v^{\beta}+av^{\alpha}}\frac{1}{1+\frac{bv^{\alpha+\beta}}{v^{\beta}+av^{\alpha}}}=\sum_{n=0}^{\infty}\frac{(-b)^{n}v^{\alpha+\beta-\sigma-1}}{v^{\beta}+av^{\alpha}+bv^{\alpha+\beta}}$$

By using collaraly 3.4, then the collaraly 3.5 can be proved.

Chapter Four

Some Applications of Fractional Sumudu Integral transform.

4. Some Applications of the Fractional Sumudu Integral transform.

4.1 Introduction.

Fractional calculus is one of the most important tools for studying physical, engineering, and scientific phenomena such as position mechanics, electrochemistry, electromagnetism, viscosity persistence, biological population models, optics, and signal processing. Fractional partial differential equations are also of great importance in various scientific applications in mathematics, physics, chemistry, and engineering.
Many techniques have been developed by researchers to solve such equations, such as the variation iteration method, the finite difference method, the symmetry perturbation method, and others. From here came a new approach that combines previous methods and integral transforms such as the Sumudu transform, the Laplace transform, and the El-Zaki transform. New methods have been discovered, such as the Sumudu homotopy perturbation method, the Sumudu decomposition method, the Laplace variation iteration method, the Laplace decomposition method, the El-Zaki homotopy perturbation method, and the El-Zaki decomposition method. These methods can be implemented to solve linear and nonlinear equations (Qazza et al., 2022).

In this chapter, we will show some applications of the Sumudu transform to solve the ordinary and partial differential equation of fractional order.

4.2 Linear Fractional Differential Equations.

Definition 4.1: (Podlubny, 1999)
The general formula of the linear fractional differential equation is given by:

$$D_t^{\alpha_n} y(t) + \sum_{i=1}^{n-1} p_i(t) D_t^{\alpha_{n-\varepsilon}} y(t) + p_n(t) y(t) = \varphi(t). \tag{4.1}$$

where $$(0 < t < T < \infty)$$

$$\left[D_t^{\alpha_{m-1}} y(t)\right] = a_m, \qquad m = 1, \dots, n, \tag{4.2}$$

where

$$D_t^{\alpha_m} \equiv D_t^{\sigma_m} D_t^{\sigma_{m-1}} \dots . D_t^{\sigma_1},$$

$$D_t^{\alpha_{m-1}} \equiv D_t^{\sigma_{m-1}} D_t^{\sigma_{m-2}} \dots . D_t^{\sigma_1},$$

$$\alpha_m = \sum_{i=1}^{m} \sigma_i , \qquad m = (1,2, \dots, n), 0 < \sigma_i \le 1, \quad (i = 1,2, \dots . , n),$$

and $\varphi(t) \in L_1(0,T)$, where $L_1(0,T)$ denotes of Lebesgue integrable functions,

where

$$\int_0^T |\varphi(t)|dt < \infty.$$

Definition 4.2: The order of a fractional differential equation is the highest exponent (fractional derivative order) appearing in the equation. It is determined by the term with the greatest differentiation power, such as $\boldsymbol{D^{\alpha}(x)}$**,** where $\boldsymbol{\alpha}$, is the largest fractional order**.**

4.3 Existence and uniqueness of the Linear Fractional Differential Equations.

Theorem 4.1:(Podlubny, 1999)

If $\varphi(t) \in L_1(0,T)$, and the function $p_i(t)(i = 1, \dots, n)$ are continuous functions on the closed interval $[0,T]$, then linear fractional differential equation (4.1) and (4.2) has a unique solution $y(t) \in L_1(0,T)$.

Here, we will present applications of the Sumudu transform by solving the integer-order differential equations, and the fractional-order differential equations.

Example 4.1: (Bodkhe & Panchal, 2016)

Solve the fractional differential equation:

$$D_t^{\alpha} y(t) - By(t) = x(t), \qquad t > 0, \quad n-1 < \alpha < n. \tag{4.3}$$

Subject to the initial conditions

$$[D_t^{(\alpha-m)} y(t)] = a_m, \qquad (m = 1,2, \dots, n) \tag{4.4}$$

where D_t^{α} denotes the Riemann-Liouville fractional derivative.

Solution: Applying the Sumudu transform to the equation (4.3), leads to

$$S[D_t^{\alpha} y(t)] - S[By(t)] = S[x(t)],$$

$$v^{-\alpha} Y(v) - \sum_{m=1}^{n} v^{-m} \left[D_t^{(\alpha-m)} y(0)\right](v) - BY(v) = X(v)$$

$$v^{-\alpha} Y(v) - \sum_{m=1}^{n} a_m v^{-m} - BY(v) = X(v)$$

$$(v^{-\alpha} - B)Y(v) = X(v) + \sum_{m=1}^{n} a_m v^{-m}.$$

Thus

$$Y(v) = \frac{X(v)}{(v^{-\alpha} - B)} + \sum_{m=1}^{n} \frac{a_m v^{-m}}{(v^{-\alpha} - B)}$$

$$Y(v) = \frac{X(v)v^{\alpha}}{(1 - Bv^{-\alpha})} + \sum_{m=1}^{n} a_m \frac{v^{\alpha-m}}{(1 - Bv^{-\alpha})}$$

Replacing $\beta = \alpha + 1$, in first term, and in the second term $\beta = \alpha - m + 1$, we obtain

$$Y(v) = X(v)S\left[t^{\alpha}E_{\alpha,\alpha}(Bt^{\alpha})\right] + \sum_{m=1}^{n} a_m\, S\left[t^{\alpha-m}E_{\alpha,\alpha-m+1}(Bt^{\alpha})\right]$$

Now take the inverse Sumudu transform and use the convolution theorem, to obtain the solution of (4.3) as

$$y(t) = \int_{0}^{t} (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\,[B(t-\tau)^{\alpha}]x(\tau)d\tau + \sum_{m=1}^{n} a_m\, t^{\alpha-m}E_{\alpha,\alpha-m+1}(Bt^{\alpha}).$$

Example 4.2: (Bodkhe & Panchal, 2016)

Solve the following initial value problem.

$$ {}_{0}^{c}D_t^{\alpha}x(t) + ax(t) = 0, \;\; 0 < \alpha \leq 1, \tag{4.5}$$

With non-zero initial condition $x(0) = B$.

Solution:

By applying the Sumudu transform to both sides of equation (4.5), one has

$$S[{}_{0}^{c}D_t^{\alpha}x(t)](v) + aS[x(t)](v) = 0$$

$$v^{-\alpha}G(v) - x(0)v^{-\alpha} + aG(v) = 0$$

$$(v^{-\alpha} + a)G(v) = Bv^{-\alpha}$$

$$G(v) = \frac{Bv^{-\alpha}}{(v^{-\alpha} + a)}$$

$$G(v) = \frac{B}{(1 + av^{\alpha})}$$

By equation (3.31) and using collaraly 3.3

$$G(v) = BS\left[E_{\alpha,1} at^{\alpha}\right].$$

By taking inverse Sumudu transform, we obtain the solution as

$$x(t) = BE_{\alpha}(-at^{\alpha}).$$

4.4 Inhomogeneous Begley-Torvik Fractional Differential Equation.

The **Begley-Torvik** equation emerged in the 1980s (1980-1986) through the research of **P.J**. **Torvik** and **R.L. Bagley,** who were studying the behavior of viscoelastic materials such as rubber and polymers. The researchers observed that classical models failed to accurately describe damping behavior, which led them to propose using fractional derivatives to model these phenomena.

The equation is named after these two scientists who developed it, as they were the first to publish systematic research demonstrating how fractional derivatives (particularly of orders 0.5 and 1.5) could model non-Newtonian damping in mechanical systems.

Definition 4.3: The Bagley- Torvik equation is a fractional differential equation with the form:

$$aD^2y(x) + bD^{\alpha}y(x) + cy(x) = \varphi(t), \tag{4.6}$$

where

$D^2y(x)$: Classical second derivative (represents acceleration), D^{α}: Fractional derivative of order α (typically 0.5 or 1.5), $\varphi(t)$: External applied force.

Key Properties:

1. **Fractional damping modeling:** Describes the hybrid behavior of viscoelastic materials.
2. **Memory effect:** Captures the complete history of system motion.
3. **Order flexibility:** α can be adjusted to model fluid-like (0.5) or solid-like (1.5) behavior.

Practical Applications:

1. **Earthquake engineering:** Analysis of vibration damping in structures.
2. **Biomedicine:** Study of blood flow in blood vessels**.**
3. **Soft robotics:** Optimization of flexible robot movement. (Podlubny, 1999)

Example 4.3: (Hamza & Elzaki, 2015)

Solve the initial value problem of the inhomogeneous Begley-Torvik equation:

$$D^2y(x) + D^{\frac{3}{2}}y(x) + y(x) = 1 + x, \tag{4.7}$$

Subject to the initial conditions

$$y(0) = y'(0) = 1$$

Solution: Applying the Sumudu transform of both sides of equation (4.7) leads to

$$\left(\frac{G(v) - y(0) - vy'(0)}{v^2}\right) + v^{2-\alpha}\left(\frac{G(v) - y(0) - vy'(0)}{v^2}\right) + G(v) = 1 + v$$

$$\frac{1}{v^2}(G(v) - 1 - v) + \frac{v^{2-\alpha}}{v^2}(G(v) - 1 - v) + G(v) = 1 + v$$

$$(G(v) - 1 - v)\left(\frac{1}{v^2} + \frac{v^{2-\alpha}}{v^2}\right) + G(v) = 1 + v$$

$$(G(v) - 1 - v)\left(\frac{1 + v^{2-\alpha}}{v^2}\right) + G(v) = 1 + v$$

$$G(v)\left(\frac{1 + v^{2-\alpha}}{v^2}\right) - (1 + v)\left(\frac{1 + v^{2-\alpha}}{v^2}\right) + G(v) = 1 + v$$

$$G(v)\left(\frac{1 + v^{2-\alpha}}{v^2} + 1\right) - \frac{(1 + v)(1 + v^{2-\alpha})}{v^2} = 1 + v$$

$$G(v)\left(\frac{1 + v^{2-\alpha}}{v^2} + 1\right) = 1 + v + \frac{(1 + v)(1 + v^{2-\alpha})}{v^2}$$

$$G(v) = \frac{(1 + v)v^2 + (1 + v)(1 + v^{2-\alpha})}{v^2} \Big/ \left(\frac{1 + v^{2-\alpha}}{v^2} + 1\right)$$

$$G(v) = \frac{(1 + v)(1 + v^{2-\alpha} + v^2)}{v^2} \times \frac{v^2}{1 + v^{2-\alpha} + v^2}$$

Then

$$G(v) = 1 + v$$

Taking the inverse Sumudu transform, one has

$$y(x) = 1 + x.$$

Remark:

Note that in this example the solution was found using the Caputo derivative, because it is considered better than the Riemann-Liouville derivative in solving problems that contain initial conditions.

Example 4.4: (Hamza & Elzaki, 2015)

Solve this following initial value problem:

$$D^{\alpha}y(t) + y(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 - t, \tag{4.8}$$

Where

$$y(0) = 0,\ \ 0 < \alpha \leq 1. \tag{4.9}$$

Solution: By applying the Sumudu transform to both sides of equation (4.8), one has

$$S[D^{\alpha}y(t)] + S[y(t)] = S\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right] + S\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right] + S[t^2] - S[t]$$

By using

$$v^{2-\alpha}\left[\frac{G(v)}{v^2} - \frac{y(0)}{v^2}\right] + G(v) = 2v^{2-\alpha} - v^{1-\alpha} + 2v^2 - v$$

$$G(v)(v^{-\alpha} + 1) = 2v^2(v^{-\alpha} + 1) - v(v^{-\alpha} + 1)$$

$$G(v) = 2v^2 - v$$

And taking the inverse to obtain

$$y(t) = 2t^2 - t.$$

Example 4.5: (Hamza & Elzaki, 2015)

Consider the composite fractional oscillation equation

$$y''(t) - aD^{\alpha}y(t) - by(t) = 8, \qquad 1 < \alpha \leq 2, \tag{4.10}$$

With initial conditions.

$$y(0) \ = \ y'(0) = 0. \tag{4.11}$$

Solve this equation by using Sumudu transform.

Solution: Using the Sumudu transform

$$\frac{G(v)}{v^2} - av^{2-\alpha}\frac{G(v)}{v^2} - bG(v) = 8$$

$$G(v) = \frac{8v^{\alpha+2}}{v^{\alpha} - av^2 - bv^{\alpha+2}}$$

Using the collaraly 3.5, then the solution is given by:

$$y(t) = 8t^2 \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(b)^n (a)^k \binom{n+k}{k}}{\Gamma(k(2-\alpha) + 2(n+1) + 1)} t^{k(2-\alpha)+2n}.$$

Example 4.6: (Hamza & Elzaki, 2015)

Solve the following linear initial value problem

$$D^{\alpha}x(t) = x(t) + 1, \qquad 0 < \alpha \leq 1,$$

$$x(0) = 0.$$

Solution: Taking the Sumudu transform:

$$\frac{G(v)}{v^{\alpha}} = G(v) + 1$$

$$\frac{G(v)}{v^{\alpha}} - G(v) = 1$$

$$G(v)\left(\frac{1 - v^{\alpha}}{v^{\alpha}}\right) = 1$$

$$G(v) = \left(\frac{v^{\alpha}}{1 - v^{\alpha}}\right)$$

Using the collaraly 3.3, we have

$$x(t) = t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha}).$$

4.5 Application to an integral production problem.

This application will demonstrate how to use the Sumudu transform to address the production and consumption problem. This study demonstrates how a factor changes over time, taking into account losses due to consumption and ensuring that the total quantity of the product remains constant. This model can be easily applied to models of population growth, birth and death processes, and problems controlling the release of drugs and hormones.

At time $t = 0$, the unused quantity of the products is N. However, this quantity is subject to depreciation according to the function $f(t)$ for, $t \geq 0$ .To counteract this loss, a new quantity $h(t)$ is produced, ensuring that the total amount remains constant at N. To generalize this model, it is supposed that the depreciation function satisfies the following condition:

$$\int_0^\infty f(t)dt = 1. \tag{4.12}$$

Thus, the total amount lost due to depreciation in the absence of production $(h(t) = 0, t > 0)$ is given by

$$\int_0^\infty Nf(t)dt = N. \tag{4.13}$$

If there is no depreciation$(f(t) \equiv 0)$, the amount of the item produced in a minor interval $[x, x + \Delta x]$ is given by $h(x)\Delta x$.

When depreciation occurs, the amount lost later t is given by $h(x)f(t - x)\Delta x$. Therefore, the total loss due to depreciation from the start until time t is expressed using the convolution integral

$$(f * h)(t) = \int_0^\infty h(x)f(t - x)dx.$$

If both production and depreciation occur simultaneously up to time t, the net difference between production and loss must be equal to $Nf(t)$. Then,

$$\int_0^t h(t)dx - \int_0^t f(t - x)h(x)dx = Nf(t). \tag{4.14}$$

Thus, this model provides a clear mathematical framework to understand how production and depreciation interact, ensuring that the total quantity of the product remains stable over time.

To solve the production problem, let $H(v)$ and $F(v)$be the Sumudu transform of the sought production function $h(t)$ and the depreciation function $f(t)$, respectively, given by

$$vH(v) - vH(v)F(v) = NF(v), \tag{4.15}$$

Then

$$H(v) = \frac{NF(v)}{v[1 - F(v)]} \tag{4.16}$$

For $f(t) = e^{-t}$, then:

$$F(v) = \frac{1}{1 + v} \text{ ,}$$

Substituting into the equation (4.16):

$$H(v) = \frac{N(1/(1 + v))}{v\left[1 - \frac{1}{1+v}\right]} = \frac{N}{v^2}, \tag{4.17}$$

As a result:

$$\frac{N}{v} = vH(v) = S\left(\int_0^t h(\tau)d\tau\right), \tag{4.18}$$

By applying the linearity of the Sumudu transform, the integral equation follows:

$$\int_0^t h(\tau)d\tau = N\delta(t),$$

For $f(t) = e^{-t}$, the production function takes the form:

$$h(t) = \frac{e^{-t}}{t}\sum_{n=1}^{\infty} nL_{n-1}(-t), \tag{4.19}$$

Where $L_{n-1}$are the Laguerre generalized polynomials, and $h(t)$ must satisfy the condition:

$$\delta(t) = \frac{e^{-\tau}}{\tau}\sum_{n=1}^{\infty}\int_0^{\infty} \frac{e^{-\tau}}{-\tau} L_{n-1}(-\tau)d\tau, \tag{4.20}$$

Thus, if a notion of a generalized derivative for $\delta(t)$ exists, then $h(t)$, as defined in the last equation, is most likely a candidate (F. B. M. Belgacem et al., 2003).

4.6 Application of the Iterative Method for Time and Space Fractional Partial Derivative Equations (PDEs).

One of the well-known methods for solving fractional-order partial differential equations using the Sumudu transform is the iterative method. The way it works is to reformulate the equation after applying the Sumudu transform to it. The principle of gradual convergence is used to reach the general solution. Through this method, the first solution is found, usually zero or an initial condition. This approach is used to solve the diffusion equation, the fractional wave equation, and the fractional heat equation.

Definition 4.4:(Kumar & Daftardar-Gejji., 2019)

Let $f(x,t) \in R$, then the Caputo time-fractional derivative operator of order $\gamma > 0, m-1 < \gamma < m, n \in N$ of the function $f(x,t)$, given by

$$\frac{\partial^\gamma f(x,t)}{\partial t^\gamma} = I_t^{m-\gamma}\left[\frac{\partial^m f(x,t)}{\partial t^m}\right],$$

$$= \begin{cases} \frac{1}{\Gamma(m-\gamma)}\int_0^t (t-\tau)^{m-\gamma-1}\frac{\partial^m f(x,\tau)}{\partial t^m}d\tau, & m-1<\gamma<m, \\ \frac{\partial^m f(x,t)}{\partial t^m}, & \gamma = m. \end{cases} \quad (4.21)$$

Similarly, the Caputo space fractional derivative operator of order $\sigma > 0, m-1 < \sigma < m, n \in N$of a real valued function $f(x,t)$, is

$$\frac{\partial^\sigma f(x,t)}{\partial x^\sigma} = I_t^{m-\sigma}\left[\frac{\partial^m f(x,t)}{\partial x^m}\right],$$

$$= \begin{cases} \frac{1}{\Gamma(m-\sigma)}\int_0^t (t-\tau)^{m-\sigma-1}\frac{\partial^m f(x,\tau)}{\partial x^m}d\tau, & m-1<\sigma<m, \\ \frac{\partial^m f(x,t)}{\partial x^m}, & \sigma = m. \end{cases} \quad (4.22)$$

In addition, the **Sumudu transform of the Caputo fractional derivative** of $f(x,t)$ of order $\alpha > 0$, is given by

$$\frac{\partial^\alpha f(x,t)}{\partial t^\alpha} = v^\alpha S[f(x,t)] - \sum_{n=0}^{m-1}\left[v^{-\alpha+n}\frac{\partial^n f(x,0)}{\partial t^n}\right], m-1<\alpha\le m, m\in N \quad (4.23)$$

The general formula of time and space FPE is given

$$\left.\begin{array}{r} \frac{\partial^\alpha f}{\partial t^\alpha} = F\left(x, f, \frac{\partial^\sigma f}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f}{\partial x^{l\sigma}}\right), m-1<\alpha<m \\ n-1<\sigma<n, n, m, l \in N \end{array}\right\} \quad (4.24)$$

Where $F\left(x, f, \frac{\partial^\sigma f}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f}{\partial x^{l\sigma}}\right)$ is a linear/nonlinear operator, and

$$\frac{\partial^{l\sigma} f}{\partial x^{l\sigma}} = \underbrace{\frac{\partial^\sigma}{\partial x^\sigma}\frac{\partial^\sigma}{\partial x^\sigma}\dots\dots\frac{\partial^\sigma f}{\partial x^\sigma}}_{\text{L-times}},$$

The function $f(x,t)$ is the unknown function.

With initial conditions

$$\frac{\partial^n f(x,0)}{\partial t^n} = g_n(x), \quad n = 0,1,2,\dots,m-1. \tag{4.25}$$

Applying the Sumudu transform to both sides of the equation. (4.24)

$$S[f(x,t)] = \sum_{n=0}^{m-1}\left[v^n \frac{\partial^n f(x,0)}{\partial t^n}\right] + v^\alpha\, S\left[F\left(x, f, \frac{\partial^\sigma f}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f}{\partial x^{l\sigma}}\right)\right]. \tag{4.26}$$

Taking the inverse Sumudu transform

$$f(x,t) = S^{-1}\left[\sum_{n=0}^{m-1}\left[v^n \frac{\partial^n f(x,0)}{\partial t^n}\right]\right] + S^{-1}\left[v^\alpha S\left(F\left(x, f, \frac{\partial^\sigma f}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f}{\partial x^{l\sigma}}\right)\right)\right]. \tag{4.27}$$

Equation (4.27) can be written the form

$$f(x,t) = u(x,0) + N\left(x, f, \frac{\partial^\sigma f}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f}{\partial x^{l\sigma}}\right), \tag{4.28}$$

Where

$$\left.\begin{aligned} u(x,0) &= S^{-1}\left[\sum_{n=0}^{m-1}\left[v^n \frac{\partial^n f(x,0)}{\partial t^n}\right]\right] \\ N\left(x, f, \frac{\partial^\sigma f}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f}{\partial x^{l\sigma}}\right) &= S^{-1}\left[v -^\alpha\, S\left(F\left(x, f, \frac{\partial^\sigma f}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f}{\partial x^{l\sigma}}\right)\right)\right] \end{aligned}\right\} \tag{4.29}$$

Represents the solution as an infinite series:

$$f = \sum_{i=0}^{\infty} f_i, \tag{4.30}$$

We can calculate operator N by the formula

$$\begin{aligned} N\left(x, \sum_{i=0}^{\infty} f_i, \frac{\partial^\sigma \sum_{i=0}^{\infty} f_i}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} \sum_{i=0}^{\infty} f_i}{\partial x^{l\sigma}}\right) &= N\left(x, f_0, \frac{\partial^\sigma f_0}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} f_0}{\partial x^{l\sigma}}\right) \\ &+ \sum_{j=1}^{\infty}\left(N\left(x, \sum_{i=0}^{j} f_i, \frac{\partial^\sigma \sum_{i=0}^{j} f_i}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} \sum_{i=0}^{j} f_i}{\partial x^{l\sigma}}\right)\right) \\ &- \sum_{j=1}^{\infty}\left(N\left(x, \sum_{i=0}^{j-1} f_i, \frac{\partial^\sigma \sum_{i=0}^{j-1} f_i}{\partial x^\sigma}, \dots, \frac{\partial^{l\sigma} \sum_{i=0}^{j-1} f_i}{\partial x^{l\sigma}}\right)\right). \end{aligned} \tag{4.31}$$

$$\left.\begin{aligned}
&S^{-1}\left[v^{\alpha}S\left(F\left(x,\sum_{i=0}^{\infty}f_i,\frac{\partial^{\sigma}\sum_{i=0}^{\infty}f_i}{\partial x^{\sigma}},\dots,\frac{\partial^{l\sigma}\sum_{i=0}^{\infty}f_i}{\partial x^{l\sigma}}\right)\right)\right]\\
&\qquad=S^{-1}\left[v^{\alpha}S\left(F\left(x,f_0,\frac{\partial^{\sigma}f_0}{\partial x^{\sigma}},\dots,\frac{\partial^{l\sigma}f_0}{\partial x^{l\sigma}}\right)\right)\right]\\
&+\sum_{j=1}^{\infty}S^{-1}\left[v^{\alpha}S\left(F\left(\left(x,\sum_{i=0}^{j}f_i,\frac{\partial^{\sigma}\sum_{i=0}^{j}f_i}{\partial x^{\sigma}},\dots,\frac{\partial^{l\sigma}\sum_{i=0}^{j}f_i}{\partial x^{l\sigma}}\right)\right)\right)\right]\\
&-\sum_{j=1}^{\infty}S^{-1}\left[v^{\alpha}S\left(F\left(\left(x,\sum_{i=0}^{j-1}f_i,\frac{\partial^{\sigma}\sum_{i=0}^{j-1}f_i}{\partial x^{\sigma}},\dots,\frac{\partial^{l\sigma}\sum_{i=0}^{j-1}f_i}{\partial x^{l\sigma}}\right)\right)\right)\right]
\end{aligned}\right\}\qquad(4.32)$$

Using (4.32) and (4.30) in (4.28)

$$\begin{aligned}
\sum_{i=0}^{\infty}f_i &= S^{-1}\left(\sum_{n=0}^{m-1}\left[v^{n}\frac{\partial^{n}f(x,0)}{\partial t^{n}}\right]\right)+S^{-1}\left[v^{\alpha}S\left(F\left(x,f_0,\frac{\partial^{\sigma}f_0}{\partial x^{\sigma}},\dots,\frac{\partial^{l\sigma}f_0}{\partial x^{l\sigma}}\right)\right)\right]\\
&+\sum_{j=1}^{\infty}S^{-1}\left[v^{\alpha}S\left(F\left(\left(x,\sum_{i=0}^{j}f_i,\frac{\partial^{\sigma}\sum_{i=0}^{j}f_i}{\partial x^{\sigma}},\dots,\frac{\partial^{l\sigma}\sum_{i=0}^{j}f_i}{\partial x^{l\sigma}}\right)\right)\right)\right]\\
&-S^{-1}\left[v^{\alpha}S\left(F\left(\left(x,\sum_{i=0}^{j-1}f_i,\frac{\partial^{\sigma}\sum_{i=0}^{j-1}f_i}{\partial x^{\sigma}},\dots,\frac{\partial^{l\sigma}\sum_{i=0}^{j-1}f_i}{\partial x^{l\sigma}}\right)\right)\right)\right],
\end{aligned}\qquad(4.33)$$

Then the recurrence relation is given by:

$$\left.\begin{aligned}
f_0 &= S^{-1}\left(\sum_{k=0}^{m-1}\left[v^n\frac{\partial^n f(x,0)}{\partial t^n}\right]\right)\\
f_1 &= S^{-1}\left[v^\alpha S\left(F\left(x,f_0,\frac{\partial^\sigma f_0}{\partial x^\sigma},\dots,\frac{\partial^{l\sigma} f_0}{\partial x^{l\sigma}}\right)\right)\right]\\
f_{r+1} &= S^{-1}\left[v^\alpha S\left(F\left(\left(x,\sum_{i=0}^{r} f_i,\frac{\partial^\sigma \sum_{i=0}^{r} f_i}{\partial x^\beta},\dots,\frac{\partial^{l\beta} \sum_{i=0}^{r} f_i}{\partial x^{l\beta}}\right)\right)\right)\right]\\
&\quad -S^{-1}\left[v^\alpha S\left(F\left(\left(x,\sum_{i=0}^{r-1} f_i,\frac{\partial^\sigma \sum_{i=0}^{r-1} f_i}{\partial x^\sigma},\dots,\frac{\partial^{l\sigma} \sum_{i=0}^{r-1} f_i}{\partial x^{l\sigma}}\right)\right)\right)\right]\\
&\qquad for\, r \geq 1
\end{aligned}\right\} \quad (4.34)$$

The approximate solution of equations (4.24) and (4.25) is given by $f \approx f_0 + f_{1+\cdots+}f_{r-1}$.

Example 4.7:(Kumar & Daftardar-Gejji., 2019)

Consider the time and space linear fractional Newell-Whitehead-Segel equation

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{\partial^{2\sigma} f}{\partial x^{2\sigma}} - 3f, \quad t > 0, \alpha, \sigma \in (0,1] \qquad (4.35)$$

With the initial condition

$$f(x,0) = E_\sigma(2x^\sigma). \qquad (4.36)$$

Applying the Sumudu transform

$$S\left[\frac{\partial^\alpha f}{\partial t^\alpha}\right] = S\left[\frac{\partial^{2\sigma} f}{\partial x^{2\sigma}} - 3f\right], \qquad (4.37)$$

Using equation (4.28), we obtain

$$S[f(x,t)] = f(x,0) + v^\alpha S\left[\frac{\partial^{2\sigma} f}{\partial x^{2\sigma}} - 3f\right], \qquad (4.38)$$

By taking the inverse Sumudu transform, we obtain:

$$f(x,t) = S^{-1}[f(x,0)] + S^{-1}\left(v^\alpha S\left[\frac{\partial^{2\sigma} f}{\partial x^{2\sigma}} - 3f\right]\right), \qquad (4.39)$$

By the recurrence relation (4.34), then

$$f_0 = S^{-1}[f(x,0)] = E_\sigma(2x^\sigma),$$

$$f_1 = S^{-1}\left(v^{\alpha} S\left[\frac{\partial^{2\sigma} f_0}{\partial x^{2\sigma}} - 3f_0\right]\right)$$

$$\frac{\partial^{2\sigma} f_0}{\partial x^{2\sigma}} = \frac{\partial^{2\sigma} E_\sigma(2x^\sigma)}{\partial x^{2\sigma}} = 2^2 E_\sigma(2x^\sigma) = 4E_\sigma(2x^\sigma).$$

Then

$$f_1 = S^{-1}(v^{\alpha} S[4E_\sigma(2x^\sigma) - 3E_\sigma(2x^\sigma)]) = S^{-1}(v^{\alpha} S[E_\sigma(2x^\sigma)])$$

$$S[E_\sigma(2x^\sigma)] = E_\sigma(2x^\sigma), \text{ constant function}$$

$$f_1 = S^{-1}\big(v^{\alpha} E_\sigma(2x^\sigma)\big) = \frac{E_\sigma(2x^\sigma) t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$f_2 = S^{-1}\left(v^{\alpha} S\left[\frac{\partial^{2\sigma}(f_0+f_1)}{\partial x^{2\sigma}} - 3(f_0+f_1)\right]\right) - f_1,$$

In the same way

$$f_2 = \frac{E_\sigma(2x^\sigma) t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$$f_3 = \frac{E_\sigma(2x^\sigma) t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

$$\vdots$$

$$f_r = \frac{E_\sigma(2x^\sigma) t^{r\alpha}}{\Gamma(r\alpha+1)}. \tag{4.40}$$

Then

$$f(x,t) = f_0 + f_1 + f_2 \ldots$$

$$f(x,t) = E_\sigma(2x^\sigma) + \frac{E_\sigma(2x^\sigma) t^{\alpha}}{\Gamma(\alpha+1)} + \frac{E_\sigma(2x^\sigma) t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{E_\sigma(2x^\sigma) t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots$$

$$f(x,t) = E_\sigma(2x^\sigma)\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right)$$

$$f(x,t) = E_\sigma(2x^\sigma). E_\alpha(x^\alpha).$$

Example 4.8:(Kumar & Daftardar-Gejji., 2019)

Solve initial condition of the linear fractional diffusion equation:

$$\frac{\partial^{\alpha} f}{\partial t^{\alpha}} = c\frac{\partial^{\sigma} f}{\partial x^{\sigma}}, t > 0, \alpha, \sigma \in (0,1]. \tag{4.41}$$

Where

$$f(x,0) = A + Bx^{\sigma}, \quad a,b \in R. \tag{4.42}$$

Solution: Applying the Sumudu transform of equation (4.41) leads to

$$S\left[\frac{\partial^{\alpha} f}{\partial t^{\alpha}}\right] = cS\left[\frac{\partial^{\sigma} f}{\partial x^{\sigma}}\right],$$

By the property of the Sumudu transform (4.23)

$$S[f(x,t)] = f(x,0) + cv^{\alpha}\left[\frac{\partial^{\sigma} f}{\partial x^{\sigma}}\right]. \tag{4.43}$$

The inverse Sumudu transform

$$f(x,t) = S^{-1}[f(x,0)] + cS^{-1}\left(v^{\alpha}\left[\frac{\partial^{\sigma} f}{\partial x^{\sigma}}\right]\right), \tag{4.44}$$

By the recurrence relation (4.34), then

$$f_0 = S^{-1}[f(x,0)] = A + Bx^{\sigma},$$

$$f_1 = cS^{-1}\left(v^{\alpha}\left[\frac{\partial^{\sigma} f_0}{\partial x^{\sigma}}\right]\right)$$

$$\frac{\partial^{\sigma} f_0}{\partial x^{\sigma}} = \frac{\partial^{\sigma}}{\partial x^{\sigma}}(A) + \frac{\partial^{\sigma}}{\partial x^{\sigma}}(Bx^{\sigma}) = 0 + \frac{\partial^{\sigma}}{\partial x^{\sigma}}(Bx^{\sigma})$$

By using a derivative of the fractional power function

$$\frac{\partial^{\sigma} x^{\mu}}{\partial x^{\sigma}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\sigma+1)} x^{\mu-\sigma}.$$

Let $\mu = \sigma$,

$$\frac{\partial^{\sigma} x^{\sigma}}{\partial x^{\sigma}} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\sigma+1)} x^{\sigma-\sigma} = \Gamma(\sigma+1)$$

Then

$$\frac{\partial^{\sigma} f_0}{\partial x^{\sigma}} = 0 + B\Gamma(\sigma+1) = B\Gamma(\sigma+1),$$

$$f_1 = cS^{-1}\big(v^{\alpha} B\Gamma(\sigma+1)\big) = \frac{cB\Gamma(\sigma+1)t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$f_2 = cS^{-1}\left(v^{\alpha}\left[\frac{\partial^{\sigma} f_1}{\partial x^{\sigma}}\right]\right)$$

$$\frac{\partial^{\sigma} f_1}{\partial x^{\sigma}} = \frac{\partial^{\sigma}}{\partial x^{\sigma}}\left(\frac{cB\Gamma(\sigma+1)t^{\alpha}}{\Gamma(\alpha+1)}\right) = 0$$

Then

$$f_2 = 0, f_3 = 0, \dots \dots$$

$$f_i = 0, i \geq 2 \tag{4.45}$$

Then the solution of equation (4.41) is given by

$$f(x,t) = A + Bx^{\sigma} + \frac{cB\Gamma(\sigma+1)t^{\alpha}}{\Gamma(\alpha+1)}.$$

Chapter Fife

Conclusions and future work.

5. Conclusions and future work

5.1 Conclusions.

In this dissertation, we studied the Sumudu fractional transform and its relationship with other well-known integral transforms, such as Laplace, Fourier, and Mellin transforms. We also presented some of its applications and methods for solving some differential equations of fractional-order.

In the **first chapter**, we introduced the concept of the Sumudu transform and studied some of its fundamental properties, such as linearity, unitary, convolution and the shitting property. Additionally, we described some important theorems related to this integral transform and included examples to illustrate the mechanism of this transform.

In the **second chapter**, we studied the relationship between the Sumudu transform and other well-known integral transforms like Laplace, Fourier, and Mellin transforms, as well as how to derive this transform from other well-known transforms. We also highlight some advantages of Sumudu transform over Laplace transform, as it becomes clear that Sumudu transform has some properties, which make it flexible in dealing with differential equations and make it more applicable in other effective physical and engineering fields.

In the **third chapter,** we presented an inference to the fractional calculus and some of its famous definitions, such as the definition of the Riemann-Liouville fractional derivative and the fractional Caputo derivative. We presented some of their properties and illustrative examples. We also studied some important and special functions in fractional calculus, such as the Beta function, Gamma function, and Mittag-Leffler function. Additionally, we studied the concept of the fractional Sumudu transform and presented some of its basic properties, such as linearity, integration, and shifting. The fractional Sumudu transform was also defined using the Riemann-Liouville derivative and Caputo derivative.

In the **fourth chapter**, we studied some applications of Sumudu transform in solving ordinary and partial differential equations. We presented some examples of linear differential equations of fractional order and how to solve them using Sumudu transform. Moreover, we studied an application of the ordinary Sumudu transform in determining population growth, as well as loss and production rates for any product using the convolution theorem.

5.2 Future work.

As future work, we plan to solve nonlinear and complex ordinary and partial differential equations of fractional order by decomposing the fractional Sumudu transform with some powerful methods, such as the Adomian decomposition method, the homotopy perturbation method or the modified Variational iteration method.

Also, studying the three-dimensional Sumudu transform. In addition, the most important properties, and how to apply it practically in various applied fields such as applied mathematics, physics, and engineering.

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