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Approximate Solutions of Different Classes of Integral Equations by using Bernstein Polynomials.

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Declaration

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Abstract

Most physical problems are modelled using differential equations, whether linear or nonlinear. Due to the difficulty of solving them directly, they are often transformed into equivalent Integral Equations (IEs), which can be solved using more effective and simple methods. This dissertation aims to obtain approximate solutions for different classes of linear IEs of both first and second kinds and with continuous and weakly singular kernels. Namely, we numerically solve IEs such as: Volterra IEs, Fredholm IEs, Volterra-Fredholm IEs, mixed Volterra-Fredholm IEs, Abel's IEs, generalized Abel's IEs, and main generalized Abel's IEs. In this dissertation, the unknown function in the IEs is approximated by an expansion in terms of Bernstein polynomials. Then by using the properties of Bernstein polynomials, the IEs are reduced by a system of linear algebraic equations, which can be easily solved to determine the coefficients of the Bernstein polynomials. These coefficients are then used to construct an approximate solution for the given integral equation. This method is characterized by its simplicity and effectiveness, especially for integral equations with complex kernels. To assess its efficiency, numerous examples were provided for all the above-mentioned classes, comparing the approximate solutions to the exact solutions. The comparisons revealed that for small values of the Bernstein polynomials degree (n), the approximate solution converged to the exact solution. The obtained results demonstrate the efficiency and reliability of this method for solving integral equations. Moreover, the obtained numerical results support the analytic theorems related to the existence, convergence, and uniqueness of the approximate solution.

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Introduction

In the eighteenth century, scientists began to be explored integral equations and their importance in understanding complex natural phenomena. In 1837 Liouville, a French mathematician, studied the relationship of integral equations to differential equations in terms of the solation an integral equation of the second kind. An integral equation was first presented by Paul de Bois-Reymoud in 1888. Therefore, the two mathematicians followed Volterra, who proved that differential equations with initial conditions lead to an integral equation and called it the Volterra equation. Fredholm integral equation was proved by Swedish mathematician Fredholm that the differential equation with boundary conditions leads to an integral equation (Wazwaz, 2011b).

Integral equations play prominent and important role in various fields of modern science and many applications such as numerous problems in engineering and mechanics, including the theory of elasticity, electrostatics, fluid Dynamics, electrodynamics, Medicine, economics, game theory, and oscillation theory(Wazwaz, 2015). It is considered one of the most important mathematical tools in both pure and applied mathematics. Due to the great development in the various sciences, which increased the complexities resulting from the interactions between them and developed greatly and rapidly, scientists began to study phenomena, whether physical, chemical, or engineering. Whereas, integral equations of various types played an important role in interpreting these phenomena and finding different solutions to them (Polyanin & Manzhirov, 2008).

In the studying of solutions of integral equations, there are many methods for solving different classes of integral equations (Sharma & Goyal, 2017) and (Jerri, 1999). Since there are two methods, namely the analytical methods and the numerical methods, many different analytical methods has been presented to obtain exact and approximate solutions of integral equations we mention some of them: The Adomian decomposition method (Wazwaz, 2015), the vibrational iteration method (Wazwaz, 2011b), the successive approximations method (Wazwaz, 2015), the Laplace transform method (Sharjeel & Barakzai, 2019) and (Patil *et al*., 2022), the Series Solution method (Wazwaz, 2011b) and the Homotopy perturbation method (Wazwaz, 2011b). The numerical methods for solving different classes of integral equations such that the collection method or iterative method (Golberg, 2013) and (Allaei *et al.,* 2017), The Nystrom method (Atkinson, 1997) and (Delves & Mohamed, 1988), The Galerkin method (Kumar & Sloan, 1987) ,The Piecewise interpolation (Hanna *et al*., 2005), The Regularized collocation method(Nair & Pereverzev, 2007).

In recent years, many researchers have been successfully applied polynomials method for solving different classes integral and intgro-differential equations, typically, these methods are based on the functions approximations for polynomials. The polynomials can be separated into two distinct types, orthogonal and semi-orthogonal. We mention some of these methods: the Legendre method are solve Volterra–Fredholm integral equations (Nemati, 2015), and Abdullah and Ali are comparison of Laguerre and Touchard Polynomials for Linear Volterra Integral and Integro Differential Equations (Abdullah & Ali, 2020). The Laguerre polynomials are solve the Singular Integral and Volterra integral equation (Qasmi et al., 2023), Also in (Zarei & Noeiaghdam, 2018). The Taylor polynomials and the collocation method solved the generalized Abel's integral equations of the first and second kinds. Bernoulli polynomials method for solving integral equations with singular kernel (Mustafa & Abd-Alrazak, 2024), Boubaker polynomials (Abdullah et al., 2024) and Chebyshev polynomials of the first kind (Piessens, 2000) Other polynomials methods (Jaber & Alrammahi, 2020; Boukhelkhal & Zeghdane, 2024 and Mustafa & Abd-Alrazak, 2024) that that provide numerical solution of different classes of linear integral equations.

Objectives

1. Introductory entrance to the classifications of integral equations regarding the linearity, continuity of the kernel, and kinds.
2. Detailed overview of the Bernstein polynomials and its properties.
3. Application of Bernstein polynomials to obtain approximate solutions of different classes of linear integral equations with continuous or weakly-singular kernels of first and second kinds including: Volterra integral equation, Fredholm integral equation, Volterra-Fredholm integral equation, mixed Volterra-Fredholm integral equation, generalized Abel integral equations, main generalized Abel integral equations and Abel's integral equations (weakly singular kernel).
4. Analyse the efficiency of this numerical method in analytical manner with regard to the existence, uniqueness and convergence, and computational manner.

Methodology

In this dissertation, we will present a straightforward, convergent, and simple numerical method based upon Bernstein approximation, which is one of the most important polynomials in approximation theory. Such method facilitates a constructive proof of the Weierstrass approximation theorem. The Bernstein polynomials began to enjoy widespread use as a versatile means of intuitively constructing and manipulating geometric shapes, spurring further development of basic theory, simple and efficient recursive algorithms. (R. T. Farouki, 2012). These polynomials have been recently used in both the solution of some linear and non-linear differential equations both partial and ordinary (Bhatti & Bracken, 2007 and Bhatta & Bhatti, 2006). Also the Bernstein polynomials have been frequently used to solve some classes of integral equations of both first and second kinds (Muhammad & Ayal, 2019 and Mandal & Bhattacharya, 2007).

Dissertation Structure

There are five chapters presented in this dissertation. In chapter one, we set up some of the preliminary concepts and basic definitions such as the definition of the integral equation, some functional spaces, some inequalities, and the classification of integral equations with regard to the linearity, nature of the kernel, and kinds.

In Chapter two, we presented an overview on the properties of the Bernstein polynomials including some recurrence relations, derivatives, integral, orthonormal property, generating function, function Approximation, the sequence and Weierstrass approximation theorem of Bernstein polynomials.

In Chapter three, the Bernstein polynomials method has been applied to approximate the solutions of linear Volterra, Fredholm, Volterra-Fredholm and mixed Volterra-Fredholm integral equations. Then, we presented some theories related to existence, uniqueness and convergence theorems of the approximate solutions. Finally, we showed some numerical results and discussion of the efficiency of the method.

In Chapter four, we solved the linear generalized Abel integral equations, main generalized Abel integral equations and Abel's integral equations with both kinds by the Bernstein polynomials method. Then, we presented the convergence, the existence and uniqueness theorems of the numerical process. For verification purposes, we presented some numerical examples and carry out a comparison of the approximate solution against the exact solution to address the efficiency of the method. Finally, Chapter five is dedicated to the conclusion and future work.

Algorithms for the numerical solution

For the numerical computations in this dissertation, we used the software Wolfram Mathematica version 14.1, which is one of the most powerful software available. We build up algorithms for each class of integral equation. These algorithms are tabulated in Appendix A.

1 Preliminary Concepts on Integral Equation

1.1 Introduction

Integral equation is considered one of the most important essential tools in applied mathematics and physics. The aim reason for using the integral equation rather than differential equations is all the conditions specifying the initial value problem or boundary value problem for a differential equation. The dimension of the problem is reduced in this process so that, such as, a boundary value problem for a differential equation in two independent variables transforms into an integral equation involving an unknown function of only one variable This reduction of what may represent a complicated mathematical model of a physical situation into a single equation is itself an important step, there are other advantage to be gained by replacing differentiation with integration. this advantage arises because integration is a smooth process, a feature which has significant implications when approximate solutions are sought. For this reason, integral equations have attracted attention for most of the last century (Ray & Sahu, 2018).

In this chapter, we will discuss the preliminary concepts of the integral equations, typical forms of the integral equation, types, classify the integral equation in terms of linearity and homogeneity, and finally we will present some of special types of kernels of integral equation.

The integral equation can be defined as the following:

Definition1.1.(Sharma & Goyal, 2017) and (Ray & Sahu, 2018) An integral equation is defined as an equation in which unknown function $\varphi(x)$ which is to be determined appears under one or more integral signs.

A standard integral equation in $\varphi(x)$ can be presented as:

$$\mu\,\varphi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\varphi(t)dt, \tag{1.1}$$

$$\int_{a(x)}^{b(x)} k(x,t)\varphi(t)dt = f(x),$$

and

$$\varphi(x) = \lambda \int_{a(x)}^{b(x)} k(x,t)[\varphi(t)]^2 dt.$$

Where the function $\varphi(x)$ is the unknown function, $a(x)$ and $b(x)$ are the limits of integration may be both variables, constants, or mixed. The functions $f(x), \mu$ and $k(x,t)$

are known for $a(x) \leq x, t \leq b(x)$, and λ is a non-zero parameter, which may be real or complex, the function $k(x,t)$ is called the kernel of integral equation.

The following two concepts play an important role in the solution integral equations.

1.2 Linearity and Homogeneity

Definition1.2. An integral equation can be called as a linear integral equation if the exponent of the unknown function $\varphi(x)$ inside the integral sign is one. The integral equation is called non-linear if the unknown function $\varphi(x)$ has exponent other than one, or if the equation contains non-linear functions of $\varphi(x)$ in equation (1.1) such as

$$\varphi^2, \varphi^3, \ldots, \sin\varphi, e^{\varphi}, \ln(\varphi+1), \ldots$$

Definition1.3.(Doha et al., 2011a) The linear integral equations can be classified as homogeneous or non-homogeneous, so if $f(x) = 0$ then it is called homogeneous, otherwise it is called nonhomogeneous integral equation.

- **Linear integral equations**

The general form of linear integral equation is given by

$$\mu\,\varphi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\varphi(t)dt \tag{1.2}$$

If the function $\mu = 0$, then equation (1.2) becomes simply

$$f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\varphi(t)dt = 0 \tag{1.3}$$

The equation (1.3) is called linear integral equation of the first kind.

If the function $\mu = \text{c} \neq 0$, then equation (1.2) becomes simply

$$c\,\varphi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\varphi(t)dt \tag{1.4}$$

The equation (1.4) is called linear integral equation of the second kind.

If μ is a function of x, then equation (1.2) is classified as linear integral equation of the third kind.

- **Nonlinear integral equations**

The general form of nonlinear integral equation is given by

$$\mu\,\varphi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)F(\varphi(t))dt \tag{1.5}$$

where $F(\varphi(t))$ is the nonlinear function inside the integral sign of $\varphi(x)$ such as $\varphi^2, \varphi^3, \ldots, \sin\varphi, e^{\varphi}$.

The nonlinear integral equation of the first kind is represented by the form:

$$f(x) = \lambda \int_{a(x)}^{b(x)} k(x,t)F(\varphi(t))dt$$

The nonlinear integral equation of the second kind is expressed in the form:

$$c\, \varphi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)F(\varphi(t))dt$$

If μ is a function of x, then equation (1.5) is classified as nonlinear integral equation of the third kind.

1.3 Classification of integral equation

Integral equations are basically classified into two main categories, namely Fredholm integral equations and Volterra integral equations, these depend mainly on the limits of integration, and however, we will present four other categories in addition to the two main categories, as the following:

1.3.1 Volterra integral equations

Volterra started working on integral equations in 1896, however, the name Volterra integral equation was first coined by Lalesco in 1908, Volterra integral equations are characterized by a variable upper limit of integration (Brunner, 2017) and (Linz, 1985). The general form of linear Volterra integral equation is given by

$$\mu\, \varphi(x) = f(x) + \lambda \int_{a}^{x} k(x,t)\varphi(t)dt \tag{1.6}$$

Where at least one of the limits of integration is a variable x and a is constant. $f(x), \mu$ and $k(x,t)$ are known functions, $\varphi(x)$ is an unknown function and λ is real or complex parameter, the $k(x,t)$ is called the kernel of integral equation.

Examples of the Volterra integral equation is given by

$$\varphi(x) = x + \frac{1}{2}\int_{a}^{x} (x-t)\varphi(t)dt$$

$$xe^{-x} = \lambda \int_{a}^{x} e^{t-x}\varphi(t)dt$$

1.3.2 Fredholm integral equation.

Fredholm was a Swedish mathematician who established the theory of integral equations in1903, Fredholm integral equations are characterized by fixed limits of integration (Atkinson, 1997).

The general form of linear Fredholm integral equation is given by

$$\mu\,\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t)dt \tag{1.7}$$

Where the limits of integration a and b are constants, $f(x), \mu$ and $k(x,t)$ are known functions, $\varphi(x)$ is an unknown function and λ is a real or complex parameter, the $k(x,t)$ is called the kernel of integral equation. The following integral equations are of Fredholm type,

$$\frac{sin(x) - xcos(x)}{x^2} = \int_0^1 sin(xt)\varphi(t)dt$$

$$\varphi(x) = x^2 + x + \int_{-1}^1 (x-t)\varphi(t)dt$$

1.3.3 Volterra-Fredholm integral equations

The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation.
The standard form of the Linear Volterra-Fredholm integral equation given by:

$$\mu\,\varphi(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\varphi(t)dt + \lambda_2 \int_a^b k_2(x,t)\varphi(t)dt$$

Where $k_1(x,t)$ and $k_2(x,t)$ are the kernels of the equation, and $\lambda_1$and λ_2 are constants parameters, x is variable and $a \le x \le b$, $f(x)$ and μ are known functions, $\varphi(x)$ is an unknown function. The following integral equations are of Volterra-Fredholm type,

$$4\,\varphi(x) = 2x^2 + x - \int_0^x (x-t)\varphi(t)dt + \int_0^1 (xt)\varphi(t)dt$$

$$\varphi(x) = x - 2e^x + e^{-x} + 1 + \int_0^x te^x\varphi(t)dt + \int_0^1 e^{(x-t)}\varphi(t)dt$$

1.3.4 Mixed Volterra-Fredholm integral equations

The mixed Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation.
The general form of the linear mixed Volterra-Fredholm integral equation given by:

$$\mu\,\varphi(x) = f(x) + \lambda \int_a^x \int_a^b k(y,t)\varphi(t)dt\,dy$$

Where $k(y,t)$ is the kernel of the equation, and λ is a constant parameter, $x\ and\ y$ are variables and $a \le x \le b\ and\ a \le y \le b$, $f(x)$ and μ are known functions, $\varphi(x)$ is unknown function.
Examples of the mixed Volterra-Fredholm integral equation are given by

$$\varphi(x) = x + \frac{17}{2}x^2 + \int_0^x \int_0^1 (y-t)\varphi(t)dt\,dy$$

$$\varphi(x) = cos(x) + sin(x) - x^2 + \frac{\pi x}{2} + \int_0^x \int_0^{\pi/2} (y-t)\varphi(t)dt\,dy$$

1.3.5 Singular integral equations

The general form of linear integral equation is given by

$$\mu\,\varphi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\varphi(t)dt \qquad (1.8)$$

Is called singular integral equation if one of the limits of integration $a(x)$, $b(x)$ or both are infinite, Moreover, the equation (1.8) is also called singular integral equation if the kernel $k(x,t)$ of the equation becomes infinite at one or more points in the interval of integration. (Ray & Sahu, 2018).

We present examples of singular integral equations

- **Abel's integral equation**

  Abel in 1823 investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravity. The particle takes the time $\varphi(t)$ to move from the highest point of vertical height t to the lowest point 0 on the curve. The Abel's problem is derived to find the equation of that curve (Wazwaz, 2011b).

  Abel derived the equation of motion of the sliding particle along a smooth curve by the singular integral equation

$$f(x) = \int_0^x k(x,t)\varphi(t)dt \qquad (1.9)$$

Or

$$\varphi(x) = f(x) + \int_0^x k(x,t)\varphi(t)dt \qquad (1.10)$$

  where $f(x)$ is a known function, and $\varphi(x)$ is an unknown function. We can call that equation (1.9) is Abel's integral equation of the first kind, and equation (1.10) is Abel's integral equation of the second kind. The kernel $k(x,t)$ in Abel's integral equation (1.9) and equation (1.10) is given by

$$k(x,t) = \frac{1}{\sqrt{x-t}}$$

  Where $k(x,t)$ becoming infinite, as $x \to t$ (Wazwaz, 2015).

  Examples of Abel's integral equation are given by

$$2\pi\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt$$

$$\varphi(x) = sin(x) + \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt$$

- **The generalized Abel's integral equation**

Abel introduced the more general singular integral equation which is given by

$$f(x) = \int_0^x \frac{1}{|x-t|^\rho} \varphi(t)dt\,, \quad 0 \ < \rho < 1 \tag{1.11}$$

The equation (1.11) is called Generalized Abel's integral equation, where $f(x)$ is a known function, and the exponent of the denominator of the kernel is ρ, where ρ are real constant such that $0 \ < \ \rho < 1$, Abel's integral equation defend above is a special case of the generalized equation where $\rho = \frac{1}{2}$ (Wazwaz, 2011b) and (Wazwaz, 2015).

Examples of the generalized Abel's integral equation are given by

$$\pi x = \int_0^x \frac{1}{(x-t)^{2/3}} \varphi(t)dt$$

$$\varphi(x) = 1 - 2x - \frac{32}{21}x^{2/4} + \frac{4}{3}x^{3/4} - \int_0^x \frac{1}{(x-t)^{1/4}} \varphi(t)dt$$

- **The main generalized Abel's integral equation**

The main generalized Abel's integral equation is given by

$$f(x) = \int_0^x \frac{1}{|H(x)-H(t)|^\rho} \varphi(t)dt\,, \quad 0 \ < \ \rho < 1 \tag{1.12}$$

Where $H(t)$ is strictly monotonically increasing and differentiable function in some interval $[a, b]$, and $H'(t) \neq 0$ for every t in the interval.

Examples of the main generalized Abel's integral equation are given by

$$\frac{4}{3} sin(x)^{3/4} = \int_0^x \frac{1}{(sin(x) - sin(t))^{1/4}} \varphi(t)dt$$

$$\varphi(x) = x - x^3 - \frac{3}{4}x^{4/3} - \frac{9}{20}x^{10/3} + \int_0^x \frac{1}{(x^2-t^2)^{1/3}} \varphi(t)dt$$

1.3.6 Integro-differential equations

An integro-differential equation introduced by Volterra 1900. In this equation, an unknown function $\varphi(x)$ occurs in one side as an ordinary derivative and appears on the other side under the integral sign.

The integro-differential equation is given by

$$\mu\, \varphi^{(m)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\varphi(t)dt \tag{1.13}$$

with initial condition $\varphi^{(m)}(x) = S_m;\ m \in \mathbb{N}$, where $\varphi^{(m)}(x) = \frac{d^m\varphi}{dx^n}$ and S_m is constant. We present types of the Integro-differential equations as follows:

- **Fredholm Integro-Differential Equations**

In the Fredholm integro-differential equations, both differential and integral operators appear in the same equation. The differential operator may be of first order or higher order. The Fredholm integro-differential equations come as a first kind and as a second kind as defined for the Fredholm integral equations. The Fredholm integro-differential equations can be presented as

$$\varphi^{(m)}(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t)dt\,; \quad m = 1,2,\ldots.$$

Example of the Fredholm integro-differential equation is given by

$$\varphi'''(x) = 2 + \sin(x) + \int_0^{\pi} (x-t)\varphi(t)dt\,, \varphi(0) = 1, \varphi'(0) = 0, and\ \varphi''(0) = -1$$

- **Volterra Integro-Differential Equations**

In the Volterra integro-differential equations, the unknown function $\varphi(x)$ and one or more of its derivatives such as $\varphi'(x)$, $\varphi''(x)$,... appear outside and inside the integral sign as well. The Volterra integro-differential equations come as a first kind and as a second kind as defined for the Volterra integral equations. The Volterra integro-differential equations can be presented as

$$\varphi^{(m)}(x) = f(x) + \lambda \int_a^x k(x,t)\varphi(t)dt$$

Example of the Volterra integro-differential equation is given as,

$$\varphi'(x) = -1 + \frac{1}{2}x^2 - xe^x - \int_0^x t\varphi(t)dt\,, \qquad \varphi(0) = 0$$

- **Volterra-Fredholm Integro-Differential Equations**

The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more ordinary derivatives in addition to the integral operators, the Volterra-Fredholm integro-differential equations can be presented as

$$\varphi^{(m)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\varphi(t)dt + \lambda_2 \int_a^b k_2(x,t)\varphi(t)dt$$

Example of the Volterra-Fredholm integro-differential equation is given by

$$\varphi'''(x) = 2\sin(x) - x - 3 + \int_0^x (x-t)\varphi(t)dt + \int_0^{\frac{\pi}{2}} \varphi(t)dt,$$

$$\varphi(0) = \varphi'(0) = 1 \text{ and } \varphi''(0) = -1$$

1.4 Special kinds of kernels

The known bivariate function $k(x,t)$ is called the kernel of the integral equation. An integral equations can be classified according to their kernel, there are many kinds of kernels of integral equation that can be presented as the following (Kenwal, 1971) and (Zemyan, 2012)

- **Symmetric kernel**

  A kernel is represented as $k(x,t)$, A kernel is said to be symmetrical (complex symmetric) if

  $$k(x,t) = \bar{k}(x,t)$$

  where the bar represents the complex conjugate, for a real kernel is symmetric if

  $$k(x,t) = k(t,x)$$

  Example of the integral equation with symmetric kernel

  $$\varphi(x) = 2\sin(x) + \int_0^1 \frac{1}{2}(x^2 + t^2)\varphi(t)dt$$

- **Difference kernel**

  The kernel $k(x,t)$ is called difference if

  $$k(x,t) = k(x-t)$$

  This is often called the coiled –type kernel.

  Example of the integral equation with difference kernel

  $$\varphi(x) = \int_0^x e^{-(x-t)}\varphi(t)dt$$

- **Separable kernel**

  Separable kernel $k(x,t)$ is also known as a degenerate kernel, it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of y only, it can be represented as follows:

  $$k(x,t) = \sum_{i=1}^{m} l_i(x)\, g_i(t)$$

  Where $l_i(x)$ and $g_i(t)$ are linear independent function sets.

Example of the integral equation with separable kernel

$$\varphi(x) = 1 + x^2 - \int_0^1 tx\varphi(t)dt$$

- **Cauchy kernel (strong singular)**

A kernel $k(x,t)$is called cauchy kernel if the kernel is of the form

$$k(x,t) = \frac{A(x,t)}{|x-t|}$$

where $A(x,t)$ is a differentiable function of (x,t) with $A(x,t) \neq 0$

Example of the integral equation with cauchy kernel (strong singular)

$$\varphi(x) = x - x^3 + \int_0^x \frac{1}{(x-t)}\varphi(t)dt$$

- **Abel's kernel (weakly singular)**

If the kernel $k(x,t)$ is of the form

$$k(x,t) = \frac{A(x,t)}{(x-t)^r}\ , \quad 0 \leq r \leq 1,$$

where $A(x,t)$ is assumed to be several times continuously differentiable. Such integral equations containing this kernel are called Abel's integral equation.

Example of the integral equation with Abel's kernel (weakly singular)

$$\varphi(x) = sin(x) + \int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt$$

- **Continuous kernel on interval $[\boldsymbol{a}, \boldsymbol{b}]$**

A kernel $k(x,t)$ is called Continuous kernel on interval $[a,b]$ if the kernel is of the form$|k(x,t)| \leq \sigma$, where σ is a constant.

Example of the integral equation with continuous kernel

$$\varphi(x) = x + \int_0^1 cos(x+t)\varphi(t)dt$$

1.5 Basic Definitions

We define some basic definitions and spaces such as Cauchy -Schwarz inequality and other mathematical terms that we need later in this dissertation (Kreyszig, 1991); (Junghanns et al., 2021) and (Atkinson, 1997).

Definition 1.4 Vector space

A vector space (or linear space) over a field $F(\mathbb{R}\ or \mathbb{C})$ is a nonempty set X of objects $x, y, \ldots$ called vectors along with two algebraic operations:

$+: X \times X \to X, (x,y) = y + x \quad \forall x,y \in X$ (Addition of vectors)

$.: F \times X \to X, (\alpha, x) = \alpha x \quad \forall \alpha \in F, x \in X$ (Multiplication of vectors)

Definition 1.5 Normed space

A normed space is a vector space with a norm defined on it. The normed space is denoted by $(X, \|.\|)$ Where a norm on vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ if it satisfies:

1- $\|x\| \geq 0, \forall x \in X$

2- $\|x\| = 0 \Leftrightarrow x = 0$

3- $\|\alpha x\| = |\alpha| \, \|x\|$

4- $\|x + y\| \leq \|x\| + \|y\|$

Definition 1.6 complete space

Let X be a linear space with norm $\|.\|_x$. We call that sequence $f_n \in X$ is *Cauchy* if

$$\forall \varepsilon > 0 \exists N: \forall n, m \geq N \ \|f_n - f_n\|_x < \varepsilon$$

If every Cauchy sequence is convergent to some of X ,then we call that X is complete.

Definition 1.7 Inner product space

Let X be a linear space over a field $F(\mathbb{R}\, or \mathbb{C})$,then the mapping $\langle . \rangle: : X \times X \to F$ is an inner product if

1- $\langle x, x \rangle \geq 0, \forall x \in X$

2- $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

3- $\langle x, y \rangle = \overline{\langle y, x \rangle}, if\ x, y \in \mathbb{C}\ and\ \langle x, y \rangle = \langle y, x \rangle\ if\ x, y \in \mathbb{R}$

4- $\langle \alpha x + \beta y, z \rangle = \langle \alpha x, z \rangle + \langle \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

Remark 1.1 An inner product on X can be defined as a norm on X given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Definition 1.8 Hilbert space

A complete inner product space is Hilbert space.

Definition 1.9 The $C[a, b]$ spaces

Let $a, b \in \mathbb{R}$, the C$[a, b]$ is the space of all continuous real valued function f

$$\mathrm{C[a,b]} = \{f: [a, b] \to F(\mathbb{R}\, or \mathbb{C})\}$$

And the norm on $C[a, b]$ is defined as

$$\|f\|_{C[a,b]} = \|f\|_\infty = \max_{a \leq x \leq b} |f|$$

Definition 1.10 The $L^p[a, b]$ spaces

L et be f function of a real variable x on an interval $[a, b]$, for each $p \geq 1, p < \infty$, we define

$$L^p[a,b] = \{\int_a^b |f|^p dx \ < \infty\}$$

And the norm on $L^p[a,b]$ is defined as

$$\|f\|_p = \left[\int_a^b |f|^p dx\right]^{1/p}$$

We denoted the norm in $L^2[a,b]$ by $\|f\|_2$.

Cauchy -Schwarz inequality

Let X be an inner product space, then for any $x, y \in X$

$$|\langle x, y\rangle| \leq \|x\|.\|y\| \tag{1.14}$$

Where$\|x\| = \sqrt{\langle x, x\rangle}$ and $\|y\| = \sqrt{\langle y, y\rangle}$

Proof. If $y = 0$, then

$$\langle x, y\rangle = \langle x, 0\rangle = \langle x, 0w\rangle = 0\langle x, w\rangle = 0 \tag{1.15}$$

$$\|x\|.\|y\| = \|x\|.0 = 0 \tag{1.16}$$

From equation (1.15) and equation (1.16), then

$$|\langle x, y\rangle| = \|x\|.\|y\|$$

- If $y \neq 0$, then

  We consider for any $u \in \mathbb{R}$,

$$0 \leq \|x + uy\|$$

$$0 \leq \langle x + uy, x + uy\rangle$$

$$0 \leq \langle x, x\rangle - \langle uy, x\rangle - \langle x, uy\rangle + \langle uy, uy\rangle$$

$$0 \leq \|x\|^2 - u\langle y, x\rangle - u\langle x, y\rangle + u^2\langle y, y\rangle$$

$$0 \leq \|x\|^2 - u\langle x, y\rangle - u\langle x, y\rangle + u^2\|y\|^2$$

$$0 \leq \|x\|^2 - 2u\langle x, y\rangle + u^2\|y\|^2$$

  We suppose

$$u = \frac{\langle x, y\rangle}{\|y\|^2} \ \Rightarrow u^2 = \frac{\langle x, y\rangle^2}{\|y\|^4}$$

$$0 \leq \|x\|^2 - \frac{2(\langle x, y\rangle)^2}{\|y\|^2} + \frac{(\langle x, y\rangle)^2}{\|y\|^2}$$

$$0 \leq \|x\|^2 - \frac{2(\langle x, y\rangle)^2}{\|y\|^2} + \frac{(\langle x, y\rangle)^2}{\|y\|^2}$$

$$0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

$$|\langle x, y\rangle|^2 \leq \|x\|^2.\|y\|^2$$

$$|\langle x, y\rangle| \leq \|x\|.\|y\|$$

2 Overview on Bernstein Polynomials

2.1 Introduction

Bernstein polynomials were introduced by the Russian mathematician Sergei Natanovich Bernstein (1920-1944). These polynomials are one of the most important polynomials in approximation theory, they facilitate a constructive proof of the Weierstrass approximation theorem. The Bernstein form began to enjoy widespread use as a versatile means of intuitively constructing and manipulating geometric shapes, spurring further development of basic theory, simple and efficient recursive algorithms. (R. T. Farouki, 2012). In this chapter, we will define Bernstein polynomials by recursive formula then investigate some of their properties.

The Bernstein polynomials can be defined as the following:

Definition 2.1(Khan et al., 2017)**:** The general form of the Bernstein polynomials of n-th degree over interval [0,1] is defined as:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \tag{2.1}$$

where

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}, i = 0,1,2,\dots,n$$

However, the Bernstein polynomials can be defined by using the binomial theorem about $(1-x)^{n-i}$, we have:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}$$

$$B_{i,n}(x) = \binom{n}{i} x^i \left(\sum_{k=0}^{n-1} (-1)^k \binom{n}{i} \binom{n-i}{k} x^k \right)$$

$$B_{i,n}(x) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k}, i = 0,1,\dots,n$$

The Bernstein polynomials of n-th degree over interval $[a,b]$ is defined as:

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, i = 0,1,2,\dots,n$$

Example 2.1: For $n = 4$ in equation (2.1), the Bernstein polynomials basis functions over interval [0,1] are given as following and the graph of all its basis functions are shown in Figure (2.1):

$$B_{0,4}(x) = \frac{4!}{0!\,(4-0)!} x^0 (1-x)^{4-0} = \frac{4!}{4!}(1-x)^4 = (1-x)^4$$

$$B_{1,4}(x) = 4x(1-x)^3$$

$$B_{2,4}(x) = 6x^2(1-x)^2$$

$$B_{3,4}(x) = 4x^3(1-x)$$

$$B_{4,4}(x) = x^4$$

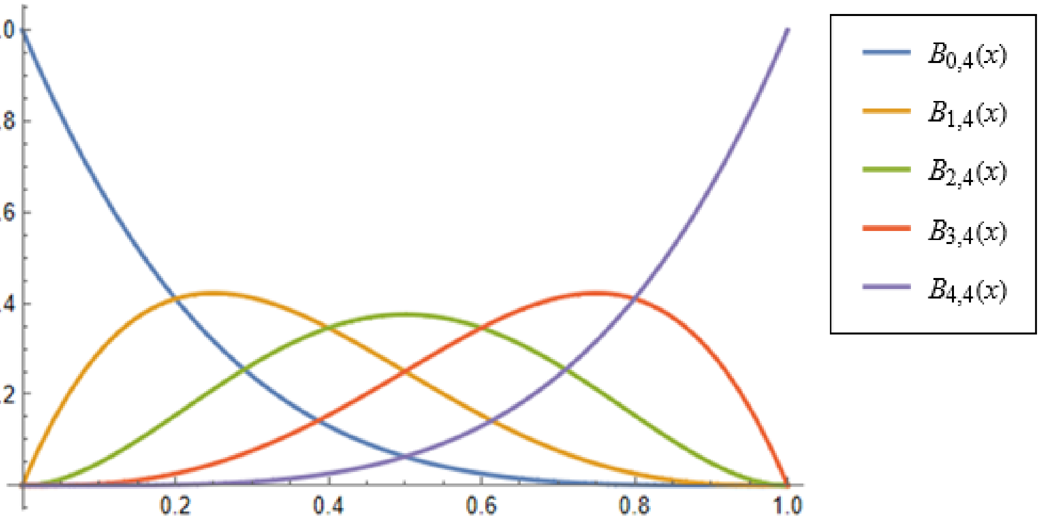

Figure (2.1): The Bernstein polynomials for $n = 4$

The following is a definition of the Bernstein polynomials using a recursive formula:

2.2 Recurrence relations of Bernstein polynomials.

Lemma 2.1. (Khan et al., 2017)**:** The Bernstein polynomials can be defined using the following recursive formula:

$$B_{i,n-1}(x) = \left(\frac{n-i}{n}\right) B_{i,n}(x) + \left(\frac{n+i}{n}\right) B_{i+1,n}(x) \tag{2.2}$$

Proof. We prove this directly from the definition of the Bernstein polynomials $B_{i,n}(x)$. We start by the right-hand side of equation (2.2)

$$\left(\frac{n-i}{n}\right) B_{i,n}(x) + \left(\frac{n+i}{n}\right) B_{i+1,n}(x)$$

$$= \left(\frac{n-i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i}$$

$$+ \left(\frac{n+i}{n}\right) \binom{n}{i+1} x^{i+1} (1-x)^{n-i-1}$$

$$= x^i (1-x)^{n-i-1} \left[\left(\frac{n-i}{n}\right) \binom{n}{i} (1-x)\right.$$

$$\left. + \left(\frac{n+i}{n}\right) \binom{n}{i+1} x\right]$$

$$= x^i (1-x)^{n-i-1} \left[\left(\frac{n-i}{n}\right) \binom{n}{i} - \left(\frac{n-i}{n}\right) \binom{n}{i} x\right.$$

$$\left. + \left(\frac{n+i}{n}\right) \binom{n}{i+1} x\right]$$

$$= x^i (1-x)^{n-i-1} \left[\left(\frac{n-i}{n}\right) \frac{n!}{i!\,(n-i)!}\right.$$

$$-\left(\frac{n-i}{n}\right)\frac{n!\,(i+1)}{(i+1)!\,(n-i)!}\,x$$

$$+\left(\frac{n+1}{n}\right)\frac{n!\,(n-i)}{(n-i)!\,(i+1)!}\,x$$

$$= x^i(1-x)^{n-i-1}\left[\frac{(n-1)!}{i!\,(n-i-1)!}\right]$$

$$= x^i(1-x)^{n-i-1}\binom{n-1}{i}$$

$$= B_{i,n-1}(x)$$

Lemma 2.2. (Khan et al., 2017)**:** The Bernstein polynomials can be defined using the following recursive formula:

$$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x) \tag{2.3}$$

Proof. We prove this by the definition of $B_{i,n}(x)$. Thus we start by the right-hand side of equation (2.3)

$$(1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x)$$

$$= (1-x)\binom{n-1}{i}x^i(1-x)^{n-i-1}$$

$$+ x\binom{n-1}{i-1}x^{i-1}(1-x)^{n-1-(i-1)}$$

$$= \binom{n-1}{i}x^i(1-x)^{n-i} + \binom{n-1}{i-1}x^i(1-x)^{n-i}$$

$$= \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right]x^i(1-x)^{n-i}$$

$$= \left[\frac{(n-1)!}{i!\,(n-i-1)!} + \frac{(n-1)!}{(i-1)!\,(n-i)!}\right]x^i(1-x)^{n-i}$$

$$= \left[\frac{(n-1)!\,(n-i)}{i!\,(n-i-1)!\,(n-i)} + \frac{(n-1)!}{(i-1)!\,(n-i)!}\right]x^i(1-x)^{n-i}$$

$$= \left[\frac{(n-1)!\,(n-i)}{i!\,(n-i)!} + \frac{(n-1)!\,i}{i!\,(n-i)!}\right]x^i(1-x)^{n-i}$$

$$= \left[\frac{n(n-1)! - i(n-1)! + i(n-1)!}{i!\,(n-i)!}\right]x^i(1-x)^{n-i}$$

$$= \binom{n}{i}x^i(1-x)^{n-i}$$

$$= B_{i,n}(x)$$

Lemma 2.3. (Khan et al., 2017)**:** The Bernstein polynomials can be defined using the following recursive formula:

$$B_{i,n}(1-x) = B_{n-i,n}(x)$$

Proof. We prove this directly from the definition of the Bernstein polynomials $B_{i,n}(x)$ given by equation (2.1), so

$$\begin{aligned} B_{i,n}(1-x) &= \binom{n}{i}(1-x)^i\,[1-(1-x)]^{n-i} \\ &= \binom{n}{i}x^{n-i}(1-x)^i \\ &= \frac{n!}{i!\,(n-i)!}\,x^{n-i}(1-x)^i \\ &= \frac{n!}{(n-i)!\,(n-(n-i))!}\,x^{n-i}(1-x)^i \\ &= \binom{n}{n-i}x^{n-i}(1-x)^i \\ &= B_{n-i,n}(x) \end{aligned}$$

2.3 Properties of the Bernstein polynomials

The Bernstein polynomials have some properties that make them a very important tool in approximation theory.

Lemma 2.4. (Khan et al., 2017)**:** The Bernstein Polynomials are all non-negative, let $f\ (x)$ be a non-negative function over the interval $[a,b]$ if $f\ (x) \geq 0$ for $x \in [a,b]$, then the Bernstein polynomials with the degree n are non-negative over the interval [0,1].

Proof. This can be proved by using definition$B_{i,n}(x)$ and mathematical induction. as

$$B_{i,n}(x) = \binom{n}{i}x^i(1-x)^{n-i}$$

For $0 \leq x \leq 1$, Let us suppose that The Bernstein Polynomials for$n = 1, i = 0$, we have

$$\begin{aligned} B_{0,1}(x) &= \binom{1}{0}x^0(1-x)^{1-0} \\ &= 1-x \geq 0 \end{aligned}$$

For $n = 1, i = 1$, we get

$$\begin{aligned} B_{1,1}(x) &= \binom{1}{1}x^1(1-x)^{1-1} \\ &= x \geq 0 \end{aligned}$$

Then the Bernstein Polynomials for $n = 1$ are non-negative.

Moreover, we prove that the Bernstein Polynomials of degree n is non-negative, by using the recursive relation of Bernstein polynomials given by equation (2.3), we put $B_{0,1}(x) = 1-x$ and $B_{1,1}(x) = x$ which are both non-negative over the interval [0,1]. Hence the Bernstein Polynomials of degree n is non-negative.

Lemma 2.5. (Doha et al., 2011b): The Bernstein Polynomials form a Partition of unity if the summation of all values of x is one, that is

$$\sum_{i=0}^{n} B_{i,n}(x) = 1$$

Proof. To prove this, we use the recursive relation of Bernstein polynomials given by equation (2.3), and rearranging the summation, we get

$$\sum_{i=0}^{n} B_{i,n}(x) = \sum_{i=0}^{n} \left[(1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x)\right]$$

$$= (1-x)\left[\sum_{i=0}^{n-1} B_{i,n-1}(x) + B_{n,n-1}(x)\right] + x\left[\sum_{i=1}^{n} B_{i-1,n-1}(x) + B_{-1,n-1}(x)\right]$$

Where we have utilized ($B_{n,n-1}(x) = B_{-1,n-1}(x) = 0$).

$$= (1-x)\left[\sum_{i=0}^{n-1} B_{i,n-1}(x)\right] + x\left[\sum_{i=1}^{n} B_{i-1,n-1}(x)\right]$$

$$= (1-x)\left[\sum_{i=0}^{n-1} B_{i,n-1}(x)\right] + x\left[\sum_{i=0}^{n-1} B_{i+1,n-1}(x)\right]$$

$$= \sum_{i=0}^{n-1} B_{i,n-1}(x)$$

Once we have established this equality, it is simple to write

$$\sum_{i=0}^{n} B_{i,n}(x) = \sum_{i=0}^{n-1} B_{i,n-1}(x) = \sum_{i=0}^{n-2} B_{i,n-2}(x) = \cdots = (1-x) + x = 1$$

Remark 2.1. The Bernstein polynomials can be expressed as:

$$B_{i,n}(1) = \delta_{i,n} = \begin{cases} 1, & if\ i = n, \\ 0, & if\ i \neq n, \end{cases}$$

and,

$$B_{i,n}(0) = \delta_{i,0} = \begin{cases} 1, & if\ i = 0, \\ 0, & if\ i \neq 0. \end{cases}$$

This proof is clear from the direct definition of the Bernstein polynomials.

Lemma 2.6. (Doha et al., 2011b): Bernstein polynomials can be written in terms of the power basis which is expressed by $\{1, x, x^2, \dots, x^n\}$

Proof. This can be proved by using the definition of Bernstein polynomials $B_{i,n}(x)$ and the binomial theorem, as follows:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}$$

$$= \binom{n}{i} x^i \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} x^k$$

$$= \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{k+i}$$

$$= \sum_{k=i}^{n} (-1)^{k-i} \binom{n}{i} \binom{n-i}{k-i} x^k$$

$$= \sum_{k=i}^{n} (-1)^{k-i} \binom{n}{k} \binom{k}{i} x^k$$

We use binomial theorem to expand $(1-x)^{n-i}$.

To show that each power basis element can be written as a linear combination of Bernstein polynomials.

$$x^i = x\left(x^{i-1}\right)$$

$$= x \sum_{k=i-1}^{n} \frac{\binom{k}{i-1}}{\binom{n}{i-1}} B_{i,n-1}(x)$$

$$= \sum_{k=i}^{n} \frac{\binom{k-1}{i-1}}{\binom{n-1}{i-1}} x\, B_{i-1,n-1}(x)$$

$$= \sum_{k=i-1}^{n} \frac{\binom{k}{i-1}}{\binom{n}{i-1}} \frac{i}{n}\, B_{i,n}(x)$$

$$= \sum_{k=i-1}^{n-1} \frac{\binom{k}{i}}{\binom{n}{i}} \frac{i}{n}\, B_{i,n}(x)$$

Lemma 2.7. (Doha et al., 2011b)**:** Let $B_{i,n}(x)$ and $B_{j,m}(x)$ be Bernstein polynomials, then

$$B_{i,n}(x) B_{j,m}(x) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j,n+m}(x), \qquad i,j = 0,1,2,\ldots,n$$

And

$$x B_{i,n}(x) = \frac{i+1}{n+1} B_{i+1,n+1}(x)$$

Proof. We prove this directly from the definition of the Bernstein polynomials $B_{i,n}(x)$ given by equation (2.1). Thus, we have

$$B_{i,n}(x) B_{j,m}(x) = \binom{n}{i} x^i (1-x)^{n-i} \binom{m}{j} x^j (1-x)^{m-j}$$

$$= \binom{n}{i}\binom{m}{j} x^{i+j} (1-x)^{n+m-(i+j)}$$

$$= \frac{\binom{n}{i}\binom{m}{j}\binom{n+m}{i+j}}{\binom{n+m}{i+j}} x^{i+j}(1-x)^{n+m-(i+j)}$$

$$= \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j,n+m}(x)$$

And

$$xB_{i,n}(x) = x\left[\binom{n}{i} x^i (1-x)^{n-i}\right]$$

$$= \binom{n}{i} x^{i+1}(1-x)^{n-i-1+1}$$

$$= \binom{n}{i} x^{i+1}(1-x)^{(n+1)-(i+1)}$$

$$= \frac{\binom{n}{i}}{\binom{n+1}{i+1}} B_{i+1,n+1}(x)$$

$$= \frac{i+1}{n+1} B_{i+1,n+1}(x)$$

2.3.1 A Matrix Representation for Bernstein Polynomials

A matrix representation is useful for the Bernstein polynomials. The linear combination of Bernstein basis functions for a given polynomials is given by (Al-Bar, 2015)

$$B(x) = A_0 B_{0,n}(x) + A_1 B_{1,n}(x) + \cdots + A_n B_{n,n}(x)$$

It is easy to write this as a dot product of two vectors

$$B(x) = \left[B_{0,n}(x) \quad B_{1,n}(x) \quad \dots \quad B_{n,n}(x)\right] \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{bmatrix} \tag{2.4}$$

The equation (2.4) can be written as:

$$B(x) = [1 \quad x \quad x^2 \quad \dots \quad x^n] \begin{bmatrix} b_{0,0} & 0 & 0 & \dots & 0 \\ b_{1,0} & b_{1,1} & 0 & \dots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_n \end{bmatrix} \tag{2.5}$$

Where $b_{i,n}$ are the coefficients of the power basis that are used to determine the respective Bernstein polynomials. We note that the matrix in this case in lower triangular matrix.

Example 2.2: The matrix representation for Bernstein polynomials in the quadratic case equation (2.5) as the following:

$$B(x) = [1 \quad x \quad x^2] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix}$$

Example 2.3: The matrix representation for Bernstein polynomials in the cubic case equation (2.5) As the following:

$$B(x) = [1 \quad x \quad x^2 \quad x^3]\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

2.3.2 Derivatives of the Bernstein Polynomials

Derivatives of the n-th degree Bernstein polynomials are polynomials of degree $(n-1)$. These derivatives can be written as a linear combination of Bernstein polynomials using the definition of Bernstein polynomials given by equation (2.1).

Lemma 2.8. (Doha et al., 2011b) The derivatives of the Bernstein polynomials over interval $[0,1]$ are defined as:

$$B'_{i,n}(x) = n\left[B_{i-1,n-1}(x) - B_{i,n-1}(x)\right] \tag{2.6}$$

Proof. We differentiate the Bernstein polynomials $B_{i,n}(x)$ defined by equation (2.1) to obtain

$$\frac{d}{dx}\left(B_{i,n}(x)\right) = \frac{d}{dx}\left(\binom{n}{i} x^i(-x)^{n-i}\right)$$

$$= \frac{d}{dx}\left(\frac{n!}{i!\,(n-i)!}\, x^i(1-x)^{n-i}\right)$$

$$= \frac{n!}{i!\,(n-i)!}\left[ix^{i-1}(1-x)^{n-1} - x^i(n-1)(1-x)^{n-i-1}\right]$$

$$= \frac{in!}{i!\,(n-i)!}\, x^{i-1}(1-x)^{n-1} - \frac{(n-1)n!}{i!\,(n-i)!} x^i(1-x)^{n-i-1}$$

$$= \frac{in!}{i(i-1)!\,(n-i)!}\, x^{i-1}(1-x)^{n-1} - \frac{(n-1)n!}{i!\,(n-i)(n-i-1)!} x^i(1-x)^{n-i-1}$$

$$= \frac{n!}{(i-1)!\,(n-i)!}\, x^{i-1}(1-x)^{n-1} - \frac{n!}{i!\,(n-i-1)!} x^i(1-x)^{n-i-1}$$

$$= n\left[\frac{(n-1)!}{(i-1)!\,(n-i)!}\, x^{i-1}(1-x)^{n-1} - \frac{(n-1)!}{i!\,(n-i-1)!} x^i(1-x)^{n-i-1}\right]$$

$$= n\left[B_{i-1,n-1}(x) - B_{i,n-1}(x)\right]$$

Remark 2.2. The derivatives of the Bernstein polynomials over interval $[a,b]$ can be defined as:

$$B'_{i,n}(x) = \frac{n}{b-a}\left[B_{i-1,n-1}(x) - B_{i,n-1}(x)\right]$$

Lemma 2.9. (Doha et al., 2011a)**:** The derivatives of the Bernstein polynomials can be presented as:

$$B'_{i,n}(x) = \frac{i - nx}{x(1-x)}\, B_{i,n}(x)$$

Proof. We differentiate the Bernstein polynomials $B_{i,n}(x)$ defined by equation (2.1) to obtain

$$\begin{aligned}
\frac{d}{dx}\left(B_{i,n}(x)\right) &= \frac{d}{dx}\left(\binom{n}{i} x^{i}(1-x)^{n-i}\right) \\
&= \binom{n}{i}\left[ix^{i-1}(1-x)^{n-1} - x^{i}(n-1)(1-x)^{n-i-1}\right] \\
&= \binom{n}{i} x^{i}(1-x)^{n-i}\left[\frac{i}{x} - \frac{n-i}{1-x}\right] \\
&= \frac{(1-x)i - x(n-i)}{x(1-x)} B_{i,n}(x) \\
&= \frac{i - xn}{x(1-x)} B_{i,n}(x)
\end{aligned}$$

2.3.3 Integral of the Bernstein Polynomials.

Integral of the n-th degree Bernstein polynomials can be written as a linear combination of Bernstein polynomials using the definition of Bernstein polynomials given by equation (2.1).

Lemma 2.10. (Doha et al., 2011a)**:** The integral of the Bernstein polynomials over interval $[0,1]$ are defined as:

$$\int_0^1 B_{i,n}(x)\, dx = \frac{1}{n+1}$$

Proof. We integrate the Bernstein polynomials $B_{i,n}(x)$ defined by equation (2.1) to obtain

$$\begin{aligned}
\int_0^1 B_{i,n}(x)\, dx &= \int_0^1 \left(\binom{n}{i} x^{i}(1-x)^{n-i}\right) dx \\
&= \binom{n}{i}\int_0^1 x^{i}(1-x)^{n-i}\, dx \\
&= \binom{n}{i}\beta(i+1, n-i+1)
\end{aligned}$$

Where β is Beta function.

$$= \binom{n}{i}\frac{\Gamma(i+1)\Gamma(\mathrm{n}-\mathrm{i}+1)}{\Gamma(n+2)}$$

Where Γ is Gamma function.

$$= \frac{n!}{i!\,(n-i)!}\frac{i!\,(n-i)!}{(n+1)!}$$

$$= \frac{n!}{(n+1)n!}$$

$$\int_0^1 B_{i,n}(x)\,dx = \frac{1}{n+1}$$

Proposition 2.1. Bernstein basis functions of the same order have the same definite integral over $[a, b]$, thus we have

$$\int_a^b B_{i,n}(x)\,dx = \frac{(b-a)}{n+1}\sum_{j=i+1}^{n+1} B_{j,n+1}(x) \tag{2.7}$$

Proposition 2.2. Bernstein basis functions of the same order have the same definite integral over the interval [a, b] can be expressed as:

$$\int_a^b B_{i,n}(x)\,dx = \frac{(b-a)}{n+1}$$

Theorem 2.1. (Doha et al., 2011a)**:** Let $I^q f(x)$ be the q-th integral of $f(x)$ with respect to x , Let q is integer number, and $P_{q-1}(x)$ is a polynomial of order $(q-1)$, then we have:

$$I^q B_{i,n}(x) = (b-a)^q \frac{n!}{(n+q)!}\sum_{j=i+q}^{n+q}\binom{j-i-1}{q-1} B_{j,n+q}(x) + P_{q-1}(x) \tag{2.8}$$

Proof. We will prove this theorem by the mathematical induction. So by using the integral of the Bernstein polynomials given by equation (2.7), we can write equation (2.8) in the following form

At $q = 1$, we get

$$IB_{i,n}(x) = (b-a)\frac{n!}{(n+1)!}\sum_{j=i+1}^{n+1}\binom{j-i-1}{0} B_{j,n+1}(x)$$

$$IB_{i,n}(x) = \frac{(b-a)}{n+1}\sum_{j=i+1}^{n+1} B_{j,n+1}(x) + P_0(x)$$

At $k = q$.

If we apply induction on q suppose that (2.8) is a true for q, we need to prove it is true at $k = q + 1$

$$I^{q+1} B_{i,n}(x) = (b-a)^{q+1}\frac{n!}{(n+q+1)!}\sum_{j=i+q+1}^{n+q+1}\binom{j-i-1}{q} B_{j,n+q+1}(x) + P_q(x)$$

$$I(I^q B_{i,n}(x)) = \int (b-a)^q \frac{n!}{(n+q)!}\sum_{j=i+q}^{n+q}\binom{j-i-1}{q-1} B_{j,n+q}(x) + P_{q-1}(x)dx$$

$$= (b-a)^q \frac{n!}{(n+q)!} \sum_{j=i+q}^{n+q} \binom{j-i-1}{q-1} \int B_{j,n+q}(x) + P_{q-1}(x)dx$$

$$= (b-a)^{q+1} \frac{n!}{(n+q+1)!} \sum_{j=i+q}^{n+q} \binom{j-i-1}{q-1} \sum_{j=i+1}^{n+q+1} B_{j,n+q+1}(x) + P_q(x)$$

If we expand the two summations $\sum_{j=i+q}^{n+q} \sum_{m=j+1}^{n+q+1}$ and collect the similar terms of $B_{j,n+q+1}(x)$ for $i+q \le j \le n+q$, we get

$$I^{q+1} B_{i,n}(x) = (b-a)^{q+1} \frac{n!}{(n+q+1)!} \sum_{j=i+q+1}^{n+q+1} \binom{j-i-1}{q} B_{j,n+q+1}(x) + P_q(x)$$

Which completes the proof.

Lemma 2.11 The q-th integral of the products Bernstein polynomials over the interval $[a, b]$ can be expressed as:

$$\int_a^b I^q B_{i,n}(x)\, B_{k,n}(x) = (b-a)^{q+1} \frac{n!}{(2n+q+1)(n+q)!} \sum_{j=i+q}^{n+q} \frac{\binom{j-i-1}{q-1}\binom{n}{k}\binom{n+q}{j}}{\binom{2n+q}{k+j}} \qquad (2.9)$$

Proof. We prove this directly from indefinite integral of $B_{i,n}(x)$ and (Theorem 2.2), thus

$$\int_a^b I^q B_{i,n}(x)\, B_{k,n}(x) = (b-a)^q \frac{n!}{(n+q)!} \sum_{j=i+q}^{n+q} \binom{j-i-1}{q-1} \int B_{j,n+q}(x) B_{k,n}(x)dx$$

$$= \frac{(b-a)^q n!}{(n+q)!} \sum_{j=i+q}^{n+q} \frac{\binom{j-i-1}{q-1}\binom{n}{k}\binom{n+q}{j}}{\binom{2n+q}{k+j}} \int B_{j+k,2n+q}(x)dx$$

$$= \frac{(b-a)^q n!}{(2n+q+1)(n+q)!} \sum_{j=i+q}^{n+q} \frac{\binom{j-i-1}{q-1}\binom{n}{k}\binom{n+q}{j}}{\binom{2n+q}{k+j}}$$

Proposition 2.3. Putting $q = 0$ in equation (2.9), we obtain

$$\int_a^b B_{i,n}(x)\, B_{k,n}(x) = \frac{(b-a)}{(2n+1)} \frac{\binom{n}{i}\binom{n}{k}}{\binom{2n}{i+k}} \quad i, k = 0,1, \dots, n \qquad (2.10)$$

2.3.4 Generating function for the Bernstein Polynomials

The generating function for the Bernstein type basis functions is a function constructed by using entire function, related to nonnegative real parameters.

The Bernstein type basis functions are defined as follows:

Definition 2.2 (Şimşek, 2014): Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let n be non-negative integer. The Bernstein type basis functions $B_i^n(x; a, b, m)$ can be defined by

$$B_i^n(x;a,b,m) = \binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^m} \quad ,i = 0,1,2,\ldots,n \tag{2.11}$$

where

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

Remark 2.3. In the special case when $n = m$ the equation (2.11) can be written as:

$$B_i^n(x;a,b) = \binom{n}{i}\left(\frac{(x-a}{b-a}\right)^i\left(\frac{b-x}{b-a}\right)^{n-i}$$

where $i = 0,1,2,\ldots,n$ and $x \in [a,b]$. One can easily get that

$$B_i^n(x) = \binom{n}{i}x^i(1-x)^{n-i}$$

where $i = 0,1,2,\ldots,n$ and $x \in [0,1]$.

Generating functions for the Bernstein type basis functions can be defined as follows.

Definition 2.3 (Şimşek, 2014)**:** Let a and b be nonnegative real parameters with $a \neq b$. Let $t \in \mathbb{C}$. Let m be a positive integer and let $x \in [a,b]$. The Bernstein type basis functions can be defined by means of the following generating function,

$$\mathrm{F}_{B,i}(x,t;a,b,m) = \sum_{n=o}^{\infty} B_i^n(x;a,b,m)\frac{t^n}{n!} \quad ,i = 0,1,2,\ldots,n$$

where $\mathrm{F}_{B,i}(x,t;a,b,m)$ denote to the hypergeometric function.

2.3.5 The Bernstein Polynomials as a Basis

Lemma 2.12. The Bernstein polynomials of n-th degree form a basis for the space of polynomials of degree less than or equal to n.

Proof. Since the Bernstein polynomials satisfy two following conditions:

1. The Bernstein polynomials are linearly independent.

We prove that as, if there exist constants $A_0, A_1, \ldots, A_n$ so that the identity

$$0 = A_0B_{0,n}(x) + A_1B_{1,n}(x) + \cdots + A_nB_{n,n}(x)$$

Hold for all x, then all the coefficients $A_i, (i = o, 1, \ldots, n)$ must be zero.

If this were true, then we could write

$$0 = A_0B_{0,n}(x) + A_1B_{1,n}(x) + \cdots + A_nB_{n,n}(x)$$

$$= A_0\sum_{i=o}^{n}(-1)^i\binom{n}{i}\binom{i}{0}x^i + A_1\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\binom{i}{1}x^i + \cdots$$

$$+ A_n\sum_{i=n}^{n}(-1)^{i-n}\binom{n}{i}\binom{i}{n}x^i$$

$$= A_0 + \left[\sum_{i=0}^{1} A_i \binom{n}{1}\binom{1}{1}\right] x^1 + \cdots + \left[\sum_{i=0}^{n} A_i \binom{n}{n}\binom{n}{n}\right] x^n$$

Since the power basis is a linearly independent set, we must have that

$$A_0 = 0 \tag{2.12}$$

$$\sum_{i=0}^{1} A_i \binom{n}{1}\binom{1}{1} = 0 \tag{2.13}$$

$$\vdots$$

$$\sum_{i=0}^{n} A_i \binom{n}{n}\binom{n}{n} = 0 \tag{2.14}$$

A_0 is clearly zero, substituting equations (2.12) and (2.13) into equation (2.14) gives $A_1 = 0$, substituting these two into equation (2.14) gives…., Then we get $A_0 = A_1 = \cdots = A_n = 0$.

2. The Bernstein polynomials are basis function for the polynomials space.

We prove that each number of the power basis $\{1, x, x^2, \ldots, x^n\}$ of the space of polynomials can be written as a linear combination of Bernstein polynomials.

The Bernstein polynomials are given by

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \tag{2.15}$$

- Case 1: For 1

We put $y = 1 - x$, then we substitute it in equation (2.15), we obtain:

$$1 = (x + 1 - x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = \sum_{i=0}^{n} B_{i,n}(x)$$

- Case 1: For x

The first derivative of equation (2.15) with respect to x are

$$n(x+y)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{i-1} y^{n-i} \tag{2.16}$$

We multiply both sides of equation (2.16) by x, and divide by n , then we put $y = 1 - x$, we get

$$x = \sum_{i=1}^{n} \frac{i}{n} B_{i,n}(x) = \sum_{i=1}^{n} \frac{\binom{i}{1}}{\binom{n}{1}} B_{i,n}(x)$$

- Case 1: For x^2

The second derivative of equation (2.15) with respect to x is

$$n(n-1)(x+y)^{n-2} = \sum_{i=1}^{n} i(i-1)\binom{n}{i} x^{i-2} y^{n-i} \tag{2.17}$$

We multiply both sides of equation (2.17) by x^2, and divide by $n(n-1)$, then we put $y = 1 - x$, we get

$$x^2 = \sum_{i=2}^{n} \frac{i(i-1)}{n(n-1)} B_{i,n}(x) = \sum_{i=1}^{n} \frac{\binom{i}{2}}{\binom{n}{2}} B_{i,n}(x)$$

However, the general case can be written as,

$$x^k = \sum_{i=k}^{n} \frac{\binom{i}{k}}{\binom{n}{k}} B_{i,n}(x), \quad k = 0,1,\dots,n$$

2.3.6 Orthonormal Bernstein Polynomials

Definition 2.4. (Singh et al., 2009) A sequence of polynomials $\{f_n(x)\}_{n=1}^{\infty}$ defined on interval $[a,b]$ are called orthogonal on this interval with respect to the weigh function $w(x) > 0$ if

$$\langle f_n(x), f_m(x)\rangle = \int_{\boldsymbol{a}}^{\boldsymbol{b}} f_n(x) f_m(x)\, w(x) = 0, \qquad n \neq m$$

Where $\langle f_n(x), f_m(x)\rangle$ is a standard notation for the inner product. The norm $\|f_n\|$ of f_n is given by

$$\|f_n\| = \sqrt{\langle f_n(x), f_m(x)\rangle} = \sqrt{\int_a^b w(x) {f_n}^2(x)}, \qquad n = m$$

Definition 2.5. A sequence of polynomials $f(x)$ and $f_m(x)$ defined on interval $[a,b]$ are called orthonormal on this interval if they are orthogonal on this interval and all have norm 1.

Then we have

$$\langle f_n(x), f_m(x)\rangle = \int_{\boldsymbol{a}}^{\boldsymbol{b}} f_n(x) f_m(x)\, w(x) = \delta_{n,m} = \begin{cases} 0, & \text{if } \ \mathrm{n} \neq \mathrm{m}, \\ 1, & \text{if } \ \mathrm{n} = \mathrm{m}. \end{cases}$$

Proposition 2.4. The Bernstein polynomials on interval $[0,1]$ form the above definitions and (Lemma 2.7.) is given by

$$\int_0^1 B_{i,n}(x) B_{j,m}(x) dx = \frac{\binom{n}{i}\binom{m}{j}}{(n-m+1)\binom{n+m}{i+j}} \neq 0$$

Hence, The Bernstein polynomials on interval $[0,1]$ are non-orthogonal.

Definition 2.6. (Bellucci, 2014) The orthonormal Bernstein polynomials are defined by analysing the resulting orthonormal polynomials $\{OB_{i,n}(x)\}_{1\le i\le n}$ by using the Gram-Schmidt process on sets of Bernstein polynomials of degree n $\{B_{i,n}(x)\}_{1\le i\le n}$. Thus we have

$$\begin{cases} \Phi_{0,n} = B_{i,n}, \;\; OB_{0,n} = \dfrac{1}{\|\Phi_{0,n}\|}\, \Phi_{0,n} \\ \Phi_{i,n} = B_{i,n} - \displaystyle\sum_{j=0}^{i-1} \langle B_{i,n}, 0B_{i,n}\rangle OB_{j,n}\,, \;\; OB_{i,n} = \dfrac{1}{\|\Phi_{i,n}\|}\, \Phi_{i,n} \end{cases}$$

Lemma 2.13.(Biazar & Ebrahimi, 2019) We can define the Gram-Schmidt process as: Let $\psi_0(x), \psi_1(x), \ldots$ be a sequence of functions on an interval $[a, b]$, satisfy merely to conditions of integrability and linear independence. The finite interval $[a, b]$ may be replaced by an infinite interval (a, ∞) or $(-\infty, \infty)$ provided that all the integrals in the process exist. We assumed that every linear combination $\psi = A_0\phi_0 + A_1\phi_1 + \cdots + A_n\phi_n$ of a finite number of ϕ's with constant coefficients $A_0, A_1, \ldots, A_n$not all zero.
Assume that

$$D_0 = \int_a^b [\phi_0(x)]^2\, dx, \qquad g_0(x) = \frac{\phi_0(x)}{\sqrt{D_0}}$$

So that $\int_a^b [g_0(x)]^2\, dx = 1$, Let

$$G_1 = \phi_0(x) - c_{1,0}g_0(x), \qquad c_{1,0} = \int_a^b \phi_1(x)g_0(x)\, dx$$

$$g_1(x) = \frac{G_1(x)}{\sqrt{D_1}}, \qquad D_0 = \int_a^b [G_1(x)]^2\, dx$$

Therefore,

$$\int_a^b G_1\, g_0 dx = 0 = \int_a^b g_1 g_0\, dx, \qquad \int_a^b {g_1}^2\, dx = 1$$

In general, let functions $g_2, g_3, \ldots$ be defined successively by the relations

$$c_{n,i} = \int_a^b \phi_n(x)g_i(x)\, dx, \qquad G_1 = \phi_n(x) - \sum_{i=0}^{n-1} c_{n,i}g_i(x),$$

$$g_n(x) = \frac{G_n(x)}{\sqrt{D_n}}, \qquad D_n = \int_a^b [G_n(x)]^2\, dx$$

Example 2.4. For $n = 5$, using the Gram-Schmidt process on $B_{i,n}$, we get the following set of orthonormal polynomials,

$$OB_{0,5}(x) = \sqrt{11}(1-x)^5$$

$$OB_{1,5}(x) = 3(1-x)^4(11x-1)$$
$$OB_{2,5}(x) = \sqrt{7}(1-x)^3(55x^2-20x+1)$$
$$OB_{3,5}(x) = \sqrt{5}(1-x)^2(165x^3-135x^2+27x-1)$$
$$OB_{4,5}(x) = \sqrt{3}(1-x)(330x^4-480x^3-216x^2+32x+1)$$
$$OB_{5,5}(x) = (462x^5-1050x^4-840x^3-280x^2+35x-1)$$

We can see from these equations that the orthonormal Bernstein polynomials are, in general of these coefficients as follows

$$\left(\sqrt{2(n-i)+1}\right)(1-x)^{n-i}$$

Lemma 2.14. Let $f(x) \in L^2[0,1]$, then the orthonormal Bernstein polynomials $OB_{i,n}(x)$ from an orthonormal system on the interval $[0,1]$ given by

$$OB_{i,n}(x) = \left(\sqrt{2(n-i)+1}\right)(1-x)^{n-i}\sum_{j=0}^{i}(-1)^j\binom{2n+1-j}{i-j}\binom{i}{j}x^{i-j} \qquad (2.18)$$

We can rewrite equation (2.18) in terms of the original non-orthonormal Bernstein basis functions as

$$OB_{i,n}(x) = \left(\sqrt{2(n-i)+1}\right)\sum_{j=0}^{i}(-1)^j\frac{\binom{2n+1-j}{i-j}\binom{i}{j}}{\binom{n-j}{i-j}}B_{i-j,n-j}(x)$$

2.3.7 Function Approximation of Bernstein Polynomials

In definition Bernstein polynomials defined form a complete basis over the interval $[a,b]$. It is easy to show that any given polynomials of degree n can be expressed in terms of linear combination of the basis functions. A function $f(x)$ defined over $[a,b]$ can be approximated by Bernstein polynomials basis functions of degree n .

Theorem 2.2. (R. Farouki & Goodman, 1996) Let f be a continuous function defined on the closed interval $[a,b]$, then

$$f(x) = \sum_{i=0}^{n} a_i\, B_{i,n}(x) \qquad (2.19)$$

where $a_i\,, i = 0,1,\dots,n$ are Bernstein coefficients and $B_{0,n}(x), B_{1,n}(x), \dots, B_{n,n}(x)$ are Bernstein basis polynomials of degree n.

Remark 2.4. Let a function $f \in L^2[0,1]$ we can write

$$f(x) = \sum_{i=0}^{n} a_i\, B_{i,n}(x)$$

The coefficients $a_{i,n}$ are given by,

$$a_i = \langle f, B_{i,n} \rangle$$

$$a_i = \int_a^b f(x) B_{i,n}(x) dx$$

2.3.8 The sequence of Bernstein Polynomials

Definition 2.7. (Phillips & Phillips, 2003) Let f be a continuous function on the interval $[a, b]$, then the sequence of Bernstein polynomials can be defined in the following form:

$$B_n(f, x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(x)$$

$$B_n(f, x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i} \tag{2.20}$$

Proposition 2.5. The sequence of Bernstein polynomials $B_n(f, x)$ is clear from equation (2.19) that for all $n \geq 1$, then we have

$$B_n(f, 0) = f(0)$$
$$B_n(f, 1) = f(1)$$

So that Bernstein polynomials interpolates f at both endpoints of the interval $[0, 1]$.

Lemma 2.15 (Qian et al., 2011)Let f is the constant function (1), then the sequence of Bernstein Polynomials $B_n(f, x)$ we obtain

$$B_n(1, x) = 1$$

Proof. We prove this directly from definition $B_n(f, x)$, we have

$$B_n(1, x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = (x + (1-x))^n = 1$$

Lemma 2.16. Let $f(t) = t$, then the sequence of Bernstein polynomials $B_n(f, x)$ are

$$B_n(t, x) = x$$

Proof. We prove this directly from definition $B_n(f, x)$ given by equation (2.19), so we obtain

$$B_n(f, x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i}$$

$$= \sum_{i=0}^{n} \frac{i}{n} \binom{n}{i} x^i (1-x)^{n-i}$$

$$= x \sum_{i=1}^{n} \binom{n-1}{i-1} x^{i-1} (1-x)^{n-i}$$

$$= x\sum_{k=0}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-k-1}$$

Thus, we get

$$B_n(f,x) = x\sum_{k=0}^{n-1}B_{k,n-1} = x$$

Lemma 2.16. The derivatives of the sequence Bernstein polynomials are given by

$$B_n'(f,x) = n\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i}\left(\frac{i}{n}-x\right)x^{i-1}(1-x)^{n-i-1}$$

Proof. We differentiate the Bernstein polynomials $B_n(f,x)$ given by equation (2.19) to obtain

$$B_n'(f,x) = \frac{d}{dx}\left[\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i}x^i(1-x)^{n-i}\right]$$

$$= \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i}ix^{i-1}(1-x)^{n-i} - \sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)\binom{n}{i}(n-i)x^i(1-x)^{n-i-1}$$

$$+[nx^{n-1}f(1) - n(1-x)^{n-1}f(0)]$$

$$= \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i}[i(1-x)-(n-i)x]x^{i-1}(1-x)^{n-i-1}$$

Thus, we get

$$B_n'(f,x) = n\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i}\left(\frac{i}{n}-x\right)x^{i-1}(1-x)^{n-i-1}$$

2.3.9 Weierstrass Approximation Theorem of Bernstein Polynomials

Theorem 2.3. (Lorentz, 2012) (Weierstrass Approximation Theorem). For each function $f(x)$ continuous on a closed interval $[a,b]$, and for each $\epsilon > 0$, there is a polynomial $P(x)$ approximating $f(x)$ uniformly with an error less than ϵ, such that

$$|f(x) - P(x)| < \epsilon \ , x \in [a,b]$$

The following theorem says that Bernstein polynomials $B_n(f,x)$ of a function f is continuous over the interval $[0,1]$ converging uniformly to f over the interval $[0,1]$.

Theorem 2.4. (Eisenberg, 1965) Let $f(x)$ be a function of $C[0,1]$, and $\epsilon > 0$ is arbitrary, there exists an integer M such that

$$|f(x) - B_n(f,x)| < \epsilon \ , x \in [0,1]$$

Proof. We prove this by using definition $B_n(f,x)$ given by equation (2.19), and we start with some important identities

$$\left(\frac{i}{n}-x\right)^2=\left(\frac{i}{n}\right)^2-2\left(\frac{i}{n}\right)x+x^2$$

We multiply each term by $\binom{n}{i}x^i(1-x)^{n-i}$ and sum from $i=0\ to\ n$, then we get

$$\sum_{i=0}^{n}\left(\frac{i}{n}-x\right)^2\binom{n}{i}x^i(1-x)^{n-i}$$

$$=\sum_{i=0}^{n}\left(\frac{i}{n}\right)^2\binom{n}{i}x^i(1-x)^{n-i}$$

$$-2\sum_{i=0}^{n}\left(\frac{i}{n}\right)x\binom{n}{i}x^i(1-x)^{n-i}+\sum_{i=0}^{n}x^2\binom{n}{i}x^i(1-x)^{n-i}$$

$$=B_n(x^2,x)-2xB_n(x,x)+x^2B_n(1,x)$$

$$=B_n(x^2,x)-2xB_n(x,x)+x^2B_n(1,x)$$

$$=\left(x^2+\frac{1}{n}x(1-x)\right)-2x(x)+x^2(1)$$

$$=\frac{1}{n}x(1-x)$$

Similarly, following the same steps, we get

$$\sum_{i=0}^{n}\left(\frac{i}{n}-x\right)^4\binom{n}{i}x^i(-x)^{n-i}$$

$$=\left[\frac{(n-1)(n-2)(n-3)x^4}{n^3}+\frac{6(n-1)(n-2)x^3}{n^3}+\frac{7(n-1)x^2}{n^3}+\frac{x}{n^3}\right]$$

$$-4x\left[\frac{(n-1)(n-2)x^3}{n^2}+\frac{3(n-1)x^2}{n^2}+\frac{x}{n^2}\right]+6x^2\left[\frac{(n-1)x^2}{n}+\frac{x}{n}\right]-4x^4+x^4$$

$$=x(1-x)\frac{(n-1)x(1-x)+1}{n^3}$$

Now, we have

$$f(x)-B_n(f,x)=\sum_{i=0}^{n}f(x)\binom{n}{i}x^i(1-x)^{n-i}-\sum_{i=0}^{n}f\left(\frac{i}{n}\right)\binom{n}{i}x^i(1-x)^{n-i}$$

$$=\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left[f(x)-f\left(\frac{i}{n}\right)\right]$$

For each $x\in[0,1]$,

$$|f(x)-B_n(f,x)|\le\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left|f(x)-f\left(\frac{i}{n}\right)\right|$$

Now, we separate the terms into two sums. For the first sum, we have $\left|\frac{i}{n}-x\right|$ is less than an integer $\delta > 0$, and the remaining terms satisfies that $\left|\frac{i}{n}-x\right| > \delta$. Now Suppose that x is a point of continuity of f . Then for any $\epsilon > 0$, there exists an integer $\delta > 0$ such that $|f(x') - f(x)| < \frac{\epsilon}{2}$ when $|x' - x| < \delta$.

Then we have $\left|\frac{i}{n}-x\right| > \delta$

$$\sum \binom{n}{i} x^i (1-x)^{n-i} \left|f(x) - f\left(\frac{i}{n}\right)\right| \leq \sum \binom{n}{i} x^i (1-x)^{n-i} \frac{\epsilon}{2}$$

$$\leq \frac{\epsilon}{2} \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i}$$

$$\leq \frac{\epsilon}{2} \sum_{i=0}^{n} B_{i,n}(x)$$

$$= \frac{\epsilon}{2}$$

Then we have $\frac{1}{\delta^2}\left(\frac{i}{n}-x\right)^2 \geq 1$

$$\sum \binom{n}{i} x^i (1-x)^{n-i} \left|f(x) - f\left(\frac{i}{n}\right)\right|$$

$$\leq \sum \binom{n}{i} x^i (1-x)^{n-i} \left|f(x) - f\left(\frac{i}{n}\right)\right|$$

$$\leq \sum_{i=0}^{n} \left(\frac{i}{n}-x\right)^2 \binom{n}{i} x^i (1-x)^{n-i} \left|f(x) - f\left(\frac{i}{n}\right)\right|$$

Since $f \in C[0,1]$, so it is bounded over the interval $[0,1]$ and we have $|f(x)| \leq M$, for some $M > 0$, hence we can denote $\left|f(x) - f\left(\frac{i}{n}\right)\right| \leq 2M$.

$$\leq \sum_{i=0}^{n} \left(\frac{i}{n}-x\right)^2 \binom{n}{i} x^i (1-x)^{n-i}\, 2M$$

$$\leq 2M \sum_{i=0}^{n} \left(\frac{i}{n}-x\right)^2 \binom{n}{i} x^i (1-x)^{n-i}$$

$$\leq 2M \frac{1}{n} x(1-x)$$

$$\leq \frac{2M}{\delta^2 n}$$

We can choose n_0 large enough so that, when $n \geq n_0$, $\frac{2M}{\delta^2 n} < \frac{\epsilon}{2}$. For such an n and the given x

$$|f(x) - B_n(f,x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This means $B_n(f,x) \to f(x)$ as $n \to \infty$ for each point x of continuity of the function f.

Theorem 2.5. (Eisenberg, 1965) Let f be a continuous differentiable function on [0,1], then the derivative sequence of Bernstein polynomials $B_n'(f,x)$, converges uniformly to $f'(x)$, such that

$$|B_n'(f,x) - f'(x)| \leq \epsilon.$$

Proof. We start the proof by applying theorem 1.7, thus we get

$$B_n'(f,x) = n\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i}\left(\frac{i}{n} - x\right)x^{i-1}(1-x)^{n-i-1}$$

Thus, for all x in $[0,1]$

$$B_n'(f,x) = n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n} - x\right)x^{i-1}(1-x)^{n-i-1}\left[f\left(\frac{i}{n}\right) - f(x)\right]$$

Suppose that x is a point of differentiability of f and a positive ϵ be given, so we write

$$\frac{f\left(\frac{i}{n}\right) - f(x)}{\frac{i}{n} - x} = f'(x) + \xi_i$$

From the assumption of differentiability of f at x, there is a positive δ such that

$$0 < |x' - x| < \delta\,, \left|\frac{f(x') - f(x)}{x' - x} - f'(x)\right| < \frac{\epsilon}{2}$$

When $0 < \left|\frac{i}{n} - x\right| < \delta$,

$$|\xi_i| = \left|\frac{f\left(\frac{i}{n}\right) - f(x)}{\frac{i}{n} - x} - f'(x)\right| < \frac{\epsilon}{2}$$

When $\left|\frac{i}{n} - x\right| > 0$, remains valid when $\frac{i}{n} = x$. Then we have

$$B_n'(f,x) = n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n} - x\right)x^{i-1}(1-x)^{n-i-1}\left[f\left(\frac{i}{n}\right) - f(x)\right]$$

$$= n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n} - x\right)x^{i-1}(1-x)^{n-i-1}\left[\left(\frac{i}{n} - x\right)f'(x) + \left(\frac{i}{n} - x\right)\xi_i\right]$$

$$= f'(x)n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n} - x\right)^2 x^{i-1}(1-x)^{n-i-1}$$

$$+n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n} - x\right)^2 x^{i-1}(1-x)^{n-i-1}\xi_i$$

$$= f'(x) + n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^2 x^{i-1}(1-x)^{n-i-1}\xi_i$$

We separate the terms into two sums. For the first sum, we have

$$\begin{aligned}\left|n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^2 x^{i-1}(1-x)^{n-i-1}\xi_i\right| &\le n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^2 x^{i-1}(1-x)^{n-i-1}|\xi_i| \\ &\le n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^2 x^{i-1}(1-x)^{n-i-1}\frac{\epsilon}{2} \\ &\le n\frac{\epsilon}{2}\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^2 x^{i-1}(1-x)^{n-i-1} \\ &\le n\frac{\epsilon}{2}\frac{1}{n}x(1-x) \\ &= \frac{\epsilon}{2}\end{aligned}$$

For the second sum, we have that $\delta \le \left|\frac{i}{n}-x\right|$ such that

$$|\xi_i| \le \left|\frac{f\left(\frac{i}{n}\right)-f(x)}{\frac{i}{n}-x}\right| + |f'(x)| \le \frac{2M}{\delta} + |f'(x)|$$

And $\delta^2 \le \left|\frac{i}{n}-x\right|^2$, then we have

$$\begin{aligned}&\delta^2\left|n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^2 x^{i-1}(1-x)^{n-i-1}\xi_i\right| \\ &\le \delta^2 n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^2 x^{i-1}(1-x)^{n-i-1}|\xi_i| \\ &\le n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^4 x^{i-1}(1-x)^{n-i-1}|\xi_i| \\ &\le n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}-x\right)^4 x^{i-1}(1-x)^{n-i-1}\left(\frac{2M}{\delta}+|f'(x)|\right) \\ &= n\frac{(3n-6)x(1-x)+1}{n^3}\left(\frac{2M}{\delta}+|f'(x)|\right) \\ &= n\frac{3}{n^2}\left(\frac{2M}{\delta}+|f'(x)|\right) = \frac{6M+3\delta|f'(x)|}{n\delta} \\ &= \frac{6M+3\delta|f'(x)|}{n\delta^3}\end{aligned}$$

For this δ, we can choose $n_0$large enough so that, when $n \ge n_0$

$$\frac{6M + 3\delta|f'(x)|}{n\delta^3} \leq \frac{\epsilon}{2}$$

For such an n and the given x

$$|B_n'(f,x) - f'(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This means $B_n'(f,x) \to f'(x)$ as $n \to \infty$ for each point x of differentiability of the function f.

3 Numerical solutions of different classes of linear integral equations by using Bernstein polynomials

3.1 Introduction

The Bernstein polynomials have been used for solving some linear and nonlinear differential equations (Bhatti & Bracken, 2007). Numerical methods based on Bernstein polynomials have been presented to solve the different classes of integral equations (Bhattacharya & Mandal, 2008);(Yilmaz & Cetin, 2017) and (Mustapha, 2022).

The aim of this chapter is to apply Bernstein polynomials methods to obtain approximate solutions of different classes of linear integral equations such as Fredholm, Volterra, Volterra-Fredholm, and mixed Volterra-Fredholm integral equations. Also, we introduce some theories related to the existence, uniqueness and convergence of the approximate solutions, and investigate the absolute error.

3.2 Bernstein Polynomials Method for Solving Linear Volterra Integral Equation

We consider the linear Volterra integral equation of the second kind given by

$$\mu\,\varphi(x) = f(x) + \lambda \int_a^x k(x,t)\varphi(t)dt \tag{3.1}$$

In this section, linear Volterra integral equation of the form (3.1) will be solved by using the Bernstein polynomials. For this, we assume that the approximate solution $\varphi(x)$ of equation (3.1) is a function approximated by Bernstein polynomials as

$$\varphi(x) \simeq \sum_{i=0}^{n} a_i\, B_{i,n}(x), \tag{3.2}$$

where $B_{i,n}(x)$ are the Bernstein polynomials of degree n defined by equation (2.19) and a_i are coefficients to be determined. Substituting equation (3.2) into equation (3.1), we obtain

$$\mu\left[\sum_{i=0}^{n} a_i B_{i,n}(x)\right] = f(x) + \lambda \int_a^x \left[k(x,t)\sum_{i=0}^{n} a_i B_{i,n}(t)\right] dt, a < x < b.$$

$$\mu\left[\sum_{i=0}^{n} a_i B_{i,n}(x)\right] - \lambda \int_a^x \left[k(x,t)\sum_{i=0}^{n} a_i B_{i,n}(t)\right] dt = f(x)$$

$$\sum_{i=0}^{n} a_i \left[\mu\, B_{i,n}(x) - \lambda \int_a^x k(x,t) B_{i,n}(t)dt\right] = f(x)$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1, \dots, n)$ and integrating both sides with respect to x from a to b, we have

$$\sum_{i=0}^{n} a_i \int_a^b \left[\mu\, B_{i,n}(x) - \lambda \int_a^x k(x,t) B_{i,n}(t) dt\right] B_{j,n}(x) dx = \int_a^b f(x) B_{j,n}(x) dx \qquad (3.3)$$

Since equation (3.3), there are three integrals, the inner integrand of the left side is a function of x and t and is integrated concerning to t from a to b. As a result, the outer integrand becomes a function of x only and integration concerning to x yields a constant. We suppose that:

$$Q_{ij} = \int_a^b \left[\mu\, B_{i,n}(x) - \lambda \int_a^x k(x,t) B_{i,n}(t) dt\right] B_{j,n}(x) dx\,, \;\; i,j = 0,1, \dots, n$$

And

$$f_j = \int_a^b f(x) B_{j,n}(x) dx$$

Then equation (3.3) can be rewritten as

$$\sum_{i=0}^{n} a_i Q_{ij} = f_j \qquad (3.4)$$

Thus, for each $j = 0,1,2, \dots, n$ we have a linear equation with $(n + 1)$ unknown coefficients $a_i, i = 0,1,2, \dots, n$. Therefore, the linear system (3.4) can be solved by any standard method to produce a_i and substituting these values back in (3.2). Thus, we get the approximate solution $\varphi(x)$ of the linear Volterra integral equation.

The Algorithm of the method

The approximate solution of the linear Volterra integral equation is obtained by an algorithm written by the Wolfram Mathematica software version 14.1, as shown in Appendix A. The algorithm consistes of the following steps,

Step 1: Input the constants a, b, c, n, λ.

Step 2: Input the functions μ, $f(x)$, $k(x,t)$, $B_{i,n}(t)$.

Step 3: Calculate the integrations Q_{ij}, f_j.

Step 4: Calculate the unknown coefficients a_i.

Step 5: Calculate the approximate solution $\varphi(x)$.

Step 6: Calculate the absolute error E.

3.2.1 Convergence and Uniqueness Solution of Linear Volterra Integral Equation

The following theorem guarantees the approximation (numerical) solution from the Bernstein polynomials method converges to the exact solution of the Volterra integral equation.

Theorem 3.1. (Uniqueness Theorem) (Mustapha, 2022) Let $f(x) \in C[a, b]$, Let $k(x, t)$ be in $C[a, b] \times [0, t_1]$. Let $Y = \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} |k(x, t)|$, suppose that there exists $\eta > 0$ such that

$$\frac{Y}{\eta} e^{\eta\,(b-a)} < 1.$$

Let $\varphi(x)$ given by equation (3.2) solution of the equation (3.1). Then the solution of the Volterra integral equation by the Bernstein polynomials method is unique.

Proof. To prove the uniqueness, let $\varphi_1(x)$ and $\varphi_2(x)$ are two different solutions for equation (3.1), thus we have

$$|\varphi_1(x) - \varphi_2(x)| = \left| f(x) + \int_a^x k(x,t)\varphi_1(t)dt - \left[f(x) + \int_a^x k(x,t)\varphi_2(t)dt \right] \right|$$

$$= \left| \int_a^x k(x,t)\varphi_1(t)dt - \left[\int_a^x k(x,t)\varphi_2(t)dt \right] \right|$$

$$= \left| \int_a^x k(x,t)\varphi_1(t)dt - \left[\int_a^x k(x,t)\varphi_2(t)dt \right] \right|$$

$$= \left| \int_a^x k(x,t)[\varphi_1(t) - \varphi_2(t)]dt \right|$$

Now, we reword the Cauchy -Schwarz inequality (1.15), we obtain

$$\leq \int_a^x |k(x,t)||\varphi_1(t) - \varphi_2(t)|dt$$

$$\leq Y \int_a^x |\varphi_1(t) - \varphi_2(t)| e^{\eta(t-a)} e^{-\eta(t-a)}\, dt$$

$$\leq \left[\frac{Y}{\eta} e^{\eta(b-a)} \right] \|\varphi_1 - \varphi_2\|, \text{ because } (t-a) \leq (b-a)$$

$$\leq \left[\frac{Y}{\eta} e^{\eta(x-a+b-x)} \right] \|\varphi_1 - \varphi_2\|$$

$$\leq \left[\frac{Y}{\eta} e^{\eta(x-a)} e^{\eta(b-x)} \right] \|\varphi_1 - \varphi_2\|$$

Since $a \leq x \leq b$, we have

$$\leq \frac{Y}{\eta} e^{\eta(x-a)} e^{\eta(b-a)} \|\varphi_1 - \varphi_2\|$$

$$|\varphi_1(x) - \varphi_2(x)| e^{-\eta(x-a)} \leq \frac{Y}{\eta} e^{\eta(b-a)} \|\varphi_1 - \varphi_2\|$$

So

$$\|\varphi_1 - \varphi_2\| \leq \frac{Y}{\eta} e^{\eta(b-a)} \|\varphi_1 - \varphi_2\|$$

$$\left(1 - \frac{Y}{\eta} e^{\eta(b-a)}\right) \|\varphi_1 - \varphi_2\| \leq 0$$

Since $\left(\frac{Y}{\eta} e^{\eta(b-a)}\right) < 1$, then we have

$$\varphi_1(x) = \varphi_2(x)$$

Hence the solution of the Volterra integral equation is unique.

Theorem 3.2. (Convergence Theorem) (SALIM et al., 2022) Let $f(x)$ be a function defined on $[a, b]$. Let $k(x, t)$ be in $[a, b] \times [0, t_1]$. Assume that the known functions $f(x)$ and $k(x, t)$ are smooth continuous and arbitrary differentiable functions in equation (3.1). Let $\varphi_{exa}(x)$ be the exact solution of equation (3.1) and $\varphi_{app}(t)$ is the approximation solution from the Bernstein polynomials method. Then

$$\lim_{n\to\infty} \left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\| = 0$$

Proof. We prove the convergence from definition of the norms, we have

$$\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\| = \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} \left|\varphi_{exa}(x) - \varphi_{app}(x)\right|$$

$$= \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} \left| f(x) + \lambda \int_a^x k(x,t)\varphi_{exa}(t)dt - \left[f(x) + \lambda \int_a^x k(x,t)\varphi_{app}(t)dt \right] \right|$$

$$= \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} \left| \lambda \int_a^x k(x,t)\varphi_{exa}(t)dt - \left[\lambda \int_a^x k(x,t)\varphi_{app}(t)dt \right] \right|$$

$$= \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} \left| \lambda \int_a^x k(x,t)\left[\varphi_{exa}(t) - \varphi_{app}(t)\right]dt \right|$$

we use Cauchy -Schwarz inequality given by equation (1.15), we obtain

$$\leq |\lambda| \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} \int_a^x |k(x,t)|\left|\varphi_{exa}(t) - \varphi_{app}(t)\right|dt$$

We suppose $Y = \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} |k(x,t)|$. Then we get

$$\leq |\lambda| Y \max_{\substack{x\in[a,b] \\ t\in[0,t_1]}} \int_a^x |k(x,t)|\left|\varphi_{exa}(t) - \varphi_{app}(t)\right|dt$$

Since $a \leq x \leq b$.

$$\leq \lambda Y(b-a)\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\|_\infty$$

We suppose $\alpha = \lambda Y \beta$, where $\beta = (b - a)$.

$$(1-\alpha)\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\|_{\infty} \leq 0$$

Therefore if $0 \leq \alpha \leq 1$ and $n \to \infty$,we have

$$\lim_{n\to\infty}\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\| = 0.$$

Definition 3.1.(Acar & Daşcıoğlu, 2017) Error is denoted by $E = \varphi_{exa}(x) - \varphi_{app}(x)$ such that $\varphi_{exa}(x)$ is the exact solution and $\varphi_{app}(x)$ is the approximated solution obtained the Bernstein polynomials, then absolute Error given by

$$|E| = \left|\varphi_{exa}(x) - \varphi_{app}(x)\right|$$

Theorem 3.3 (Ray & Sahu, 2018) Let $\varphi_{exa}(x)$ be the exact solution the Volterra integral equation in equation (3.1) and $\varphi_{app}(x) = \sum_{i=0}^{n} a_i\, B_{i,n}(x), i = 0,1,\dots,n$ be the approximation solution of equation (3.1), where $B_{i,n}(x)$ is the Bernstein polynomials of degree n. Then $\|E_n\| \to 0$, if the above approximation solution converges to the exact solution of the Volterra integral equation, when $n \to \infty$.

Proof. Consider $\varphi_{exa}(x)$ as the exact solution of the Volterra integral equation (3.1). Let $\varphi_{app}(x)$ be the approximate solution of equation (3.1), we have

$$\mu\, \varphi_{app}(x) = f(x) + \int_a^x k(x,t)\varphi_{app}(t)dt \tag{3.5}$$

Now define the error functions E_n by subtracting equation (3.5) from equation (3.1) as follows:

$$E_n = \varphi_{exa}(x) - \varphi_{app}(x)$$

We have

$$E_n = \left[\varphi_{exa}(x) - \varphi_{app}(x)\right] - \int_a^x k(x,t)\left[\varphi_{exa}(t) - \varphi_{app}(t)\right]dt$$

$$\|E_n\| = \left\|\left[\varphi_{exa}(x) - \varphi_{app}(x)\right] - \int_a^x k(x,t)\left[\varphi_{exa}(t) - \varphi_{app}(t)\right]dt\right\|$$

Now we use Cauchy -Schwarz inequality given by equation (1.15), to obtain

$$\|E_n\| \leq \left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\| - \int_0^x \|k(x,t)\|\left\|\varphi_{exa}(t) - \varphi_{app}(t)\right\|\, dt$$

Since the norm $\|k(x,t)\|$ is bound, and from Theorem (3.1) we have $\left\|\varphi_{exa}(t) - \varphi_{app}(t)\right\| \to 0$ as $n \to \infty$, Hence $\|E_n\| \to 0$ as $n \to \infty$.

3.2.2 Numerical Results and Discussion

In this section, various numerical examples are presented to illustrate the effectiveness of the method described. We compare the exact solution $\varphi_{exa}(x)$ with the approximate solution $\varphi_{app}(x)$, and we calculate the absolute error E.

Example 3.1. Find the solution linear Volterra integral equation of the first kind with the exponential kernel:

$$\int_0^x e^{(x-t)}\varphi(t)dt = \sin(x), \qquad x \in [0,1] \tag{3.6}$$

We have the exact solution (Abdullah & Ali, 2020) as

$$\varphi(x) = \cos(x) - \sin(x).$$

Using Bernstein polynomials to solve equation (3.6) with $n = 3$, we have the following:

$$\int_0^x e^{(x-t)}\varphi(t)dt = \sin(x)$$

To determine the approximate solution of equation (3.6), we have

$$\varphi_{app}(x) = \sum_{i=0}^{3} a_i\, B_{i,3}(x) \tag{3.7}$$

Substituting equation (3.7) into equation (3.6), to obtain

$$\int_0^x e^{(x-t)}\sum_{i=0}^{3} a_i\, B_{i,3}(t)dt = \sin(x)$$

$$\sum_{i=0}^{3} a_i\left[\int_0^x e^{(x-t)}B_{i,3}(t)dt\right] = \sin(x)$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,2,3)$ and integrating both sides with respect to x from 0 to 1, leads to

$$\sum_{i=0}^{3} a_i\int_0^1\left[\int_0^x e^{(x-t)}B_{i,3}(t)dt\right]B_{j,3}(x)dx = \int_0^1 \sin(x)\, B_{j,3}(x)dx \tag{3.8}$$

Now, we suppose that:

$$Q_{ij} = \int_0^1\left[\int_0^x e^{(x-t)}B_{i,3}(t)dt\right]B_{j,3}(x)dx, \;\; i,j = 0,1,2,3$$

And

$$f_j = \int_0^1 \sin(x)\, B_{j,3}(x)dx\,, j = 0,1,2,3$$

Then equation (3.8) can be rewritten as

$$\sum_{i=0}^{3} a_i Q_{ij} = f_j \tag{3.9}$$

We can calculate integrals Q_{ij} and f_j, and substitute them into equation (3.9), we get a system of linear equations

$$0.037a_0 + 0.016a_1 + 0.005a_2 + 0.001a_3 = 0.048$$
$$0.067a_0 + 0.039a_1 + 0.018a_2 + 0.005a_3 = 0.095$$
$$0.092a_0 + 0.066a_1 + 0.039a_2 + 0.015a_3 = 0.138$$
$$0.116a_0 + 0.092a_1 + 0.067a_2 + 0.037a_3 = 0.177$$

Therefore, we solve the above system of linear equation to get

$$a_0 = 0.999,\ a_1 = 0.667,\ a_2 = 0.161,\ a_3 = -0.302$$

Then, we substitute these values back in equation (3.7) to get the approximate solution of equation (3.6) for $n = 3$ as follows

$$\varphi_{app}(x) = 0.999 - 0.996x - 0.525x^2 + 0.219x^3$$

Numerical results are listed in Table 3.1 illustrate the convergence of the approximate solution $\varphi_{app}(x)$ to the exact solution $\varphi_{exa}(x)$ at different values of x over the interval $[0,1]$. Abdullah and Ali (Abdullah & Ali, 2020) solved equation (3.6) by Laguerre and Touchard polynomials with five different degrees $n = 2,3,4,5,6$, where the absolute error has an order of approximately 10^{-3}. Our absolute error is calculated by Bernstein polynomials for each value and has an order of approximately 10^{-5} when $n = 3$. The comparison of absolute error of the proposed method and those in (Abdullah & Ali, 2020), showing the Bernstein polynomials method having a higher accuracy than in (Abdullah & Ali, 2020) with $n = 3$.

Table 3.1. Numerical results and absolute errors for equation (3.6).

x	Exact solution $\varphi_{exa}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$	
		$n = 3$	$n = 5$
0	1	$1.67172575311 * 10^{-4}$	$4.626905452109 * 10^{-7}$
0.1	0.8951707486311977	$6.26733402472 * 10^{-4}$	$1.605340264054 * 10^{-7}$
0.2	0.7813972470461804	$4.92714244889 * 10^{-5}$	$1.217402052502 * 10^{-7}$
0.300000000004	0.6598162824642664	$2.91911825749 * 10^{-5}$	$8.931302963866 * 10^{-8}$
0.4	0.5316426516942345	$7.25369303941 * 10^{-5}$	$1.039665908919 * 10^{-7}$
0.5	0.39815702328616975	$4.64600097241 * 10^{-5}$	$1.122427017996 * 10^{-7}$
0.6	0.26069314151464296	$3.02696641930 * 10^{-5}$	$7.847412736294 * 10^{-8}$

x	Exact solution $\varphi_{exa}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$	
		$n = 3$	$n = 5$
0.7	0.120624500046797	$9.91327835434 * 10^{-5}$	$1.567483805808 * 10^{-7}$
0.799999999999	-0.020649381552357	$7.55841426467 * 10^{-5}$	$1.106760344898 * 10^{-7}$
0.9	-0.1617169413568188	$1.36964 * 10^{-4}$	$2.152292487567 * 10^{-7}$
1.	-0.3011686789397563	$6.33037 * 10^{-4}$	$1.877454396059 * 10^{-6}$

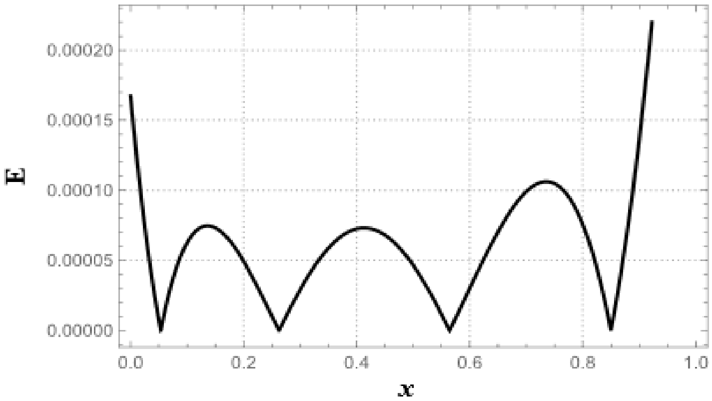

Figure 3.1(a): Absolute error for $n = 3$ of equation (3.6)

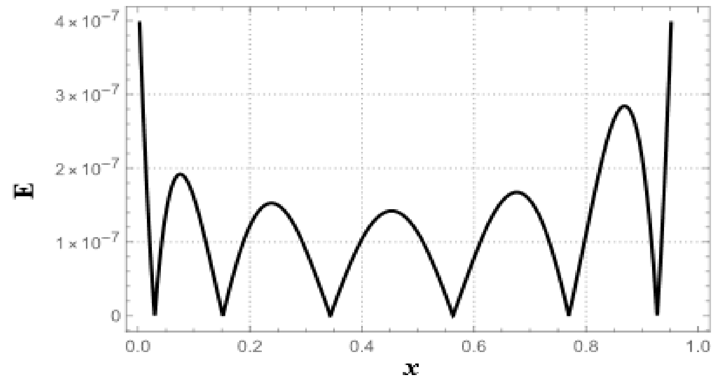

Figure 3.1(b): Absolute error for $n = 5$ of equation (3.6)

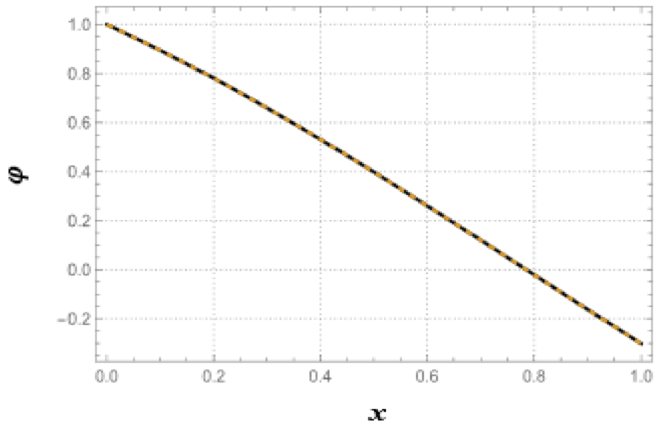

Figure 3.1(c): Exact and the approximate solutions of equation (3.6)

Example 3.2. Solve the following linear Volterra integral equation of the second kind:

$$\varphi(x) = \cos(x) + \sin(x) - \int_0^x \varphi(t)dt \tag{3.10}$$

We have the exact solution(Aggarwal et al., 2023) as:

$$\varphi(x) = \cos(x)$$

We apply the algorithm of the Bernstein polynomials method with degree $n = 4$, we obtain the linear system

$$\frac{59}{450}a_0 + \frac{29}{450}a_1 + \frac{19}{700}a_2 + \frac{2}{225}a_3 + \frac{11}{6300}a_4 = 0.228$$
$$\frac{13}{150}a_0 + \frac{263}{3150}a_1 + \frac{61}{1050}a_2 + \frac{31}{1050}a_3 + \frac{2}{225}a_4 = 0.250$$
$$\frac{127}{2100}a_0 + \frac{27}{350}a_1 + \frac{27}{350}a_2 + \frac{61}{1050}a_3 + \frac{19}{700}a_4 = 0.266$$
$$\frac{74}{1575}a_0 + \frac{193}{3150}a_1 + \frac{27}{350}a_2 + \frac{263}{3150}a_3 + \frac{29}{450}a_4 = 0.276$$
$$\frac{29}{700}a_0 + \frac{74}{1575}a_1 + \frac{127}{2100}a_2 + \frac{13}{150}a_3 + \frac{59}{450}a_4 = 0.279$$

We solve the above linear system of equation, we get

$$a_0 = 0.999,\ a_1 = 1,\ a_2 = 0.916,\ a_3 = 0.750,\ a_4 = 0.540$$

Then, we substitute these values back in equation (3.2), we get the approximate solution of equation (3.10) when $n = 4$ as follows

$$\varphi_{app}(x) = 0.999 + 0.004x - 0.503x^2 + 0.006x^3 + 0.036x^4$$

Table 3.2 and Figure 3.2(a) present some values of the exact and the approximate solutions calculated at some various values over the interval $[0,1]$. The Bernstein polynomials method is applied with $n = 4$. The absolute error is calculated for each value and has an order of approximately 10^{-6}all over $[0,1]$. Figure 3.2(b) illustrates the behaviour of the absolute error for $n = 4$. The obtained results confirm the effectiveness of the method described.

Table 3.2. The approximate and the exact solutions of equation (3.10).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	1	0.999986306004835	$1.369399516470 * 10^{-5}$
0.1	0.99500416527802	0.995009751823458	$5.586545432811 * 10^{-6}$
0.2	0.98006657784124	0.980068932316433	$2.354475191590 * 10^{-6}$
0.3000000004	0.9553364891256	0.95533255847671	$3.930648887262 * 10^{-6}$

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0.4	0.92106099400288	0.92105622651914	$4.767483744116 * 10^{-6}$
0.5	0.87758256189037	0.877582417880393	$1.440099793854 * 10^{-7}$
0.6	0.82533561490967	0.825340499219035	$4.884309357078 * 10^{-6}$
0.7	0.764842187284488	0.764846722415493	$4.535131005178 * 10^{-6}$
0.7999999999	0.696706709347165	0.696704224572061	$2.484775103783 * 10^{-6}$
0.8999999999	0.621609968270664	0.621603028012898	$6.940257765708 * 10^{-6}$
0.9999999999	0.540302305868139	0.540320040284031	$1.773441589214 * 10^{-4}$

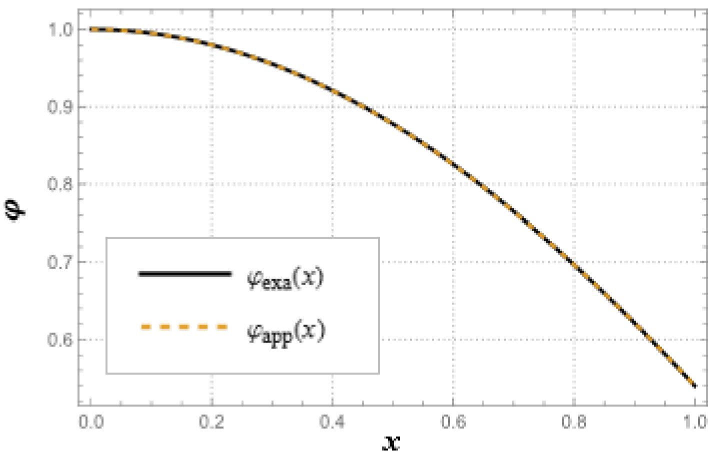

Figure 3.2(a): Exact and the approximate solutions of equation (3.10)

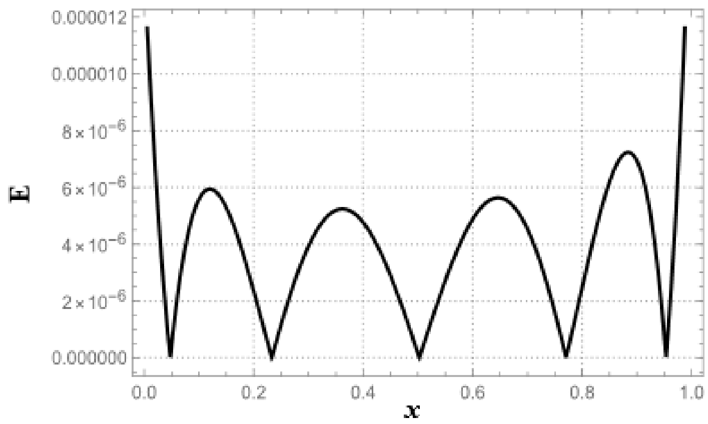

Figure 3.2(b): Absolute error for $n = 4$ of equation (3.10)

Example 3.3. Solve the following linear Volterra integral equation of the second kind:

$$\varphi(x) = \sin(x) + 2\int_0^x e^{(x-t)}\varphi(t)dt \tag{3.11}$$

Where the exact solution(Aggarwal et al., 2023) as,

$$\varphi(x) = \frac{1}{5}e^{3x} - \frac{1}{5}\cos(x) - \frac{2}{5}\sin(x)$$

Solution: We apply the algorithm of the Bernstein polynomials method with degree $n = 4$, we get the approximate solution of equation (3.11).

$$\varphi_{app}(x) = 0.007 + 0.787x + 2.350x^2 - 2.304x^3 + 3.396x^4$$

In Table 3.3, we display the exact and the approximate solutions calculated in some values of x over the interval $[0,1]$. The proposed method is applied with $n = 4$. The absolute error is also calculated for each value and has an order of approximately 10^{-3}. Figure 3.3(b) illustrates the behaviour of the absolute error for $n = 4$, and the results obtained in Table 3.3 and Figure 3.3(a) confirm that the Bernstein polynomials mothed approximates the solution in the same way throughout the interval.

Table 3.3. Exact and approximate solutions are different values of x for equation (3.11).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	0	0.007734866925290	$7.734866925290 * 10^{-3}$
0.1	0.110904295118326	0.108016059146669	$2.888235971657 * 10^{-3}$
0.2	0.247878176827878	0.246225001486675	$1.653175341202 * 10^{-3}$
0.3000000004	0.419061407070804	0.420761627098570	$1.700220027765 * 10^{-3}$
0.4	0.635578522670192	0.63817796080971	$2.599438139517 * 10^{-3}$
0.5	0.912591517131219	0.913178119121551	$5.86601990334 * 10^{-3}$
0.6	1.270719359258667	1.268618310209648	$2.101049049019 * 10^{-3}$
0.7	1.737952619951708	1.735506833923652	$2.445786028055 * 10^{-3}$
0.7999999999	2.352236370618696	2.353004081787314	$7.677111686179 * 10^{-3}$
0.8999999999	3.164955115171426	3.168422536998480	$3.467421827053 * 10^{-3}$
0.9999999999	4.245635317387062	4.237226774429098	$8.408542957964 * 10^{-3}$

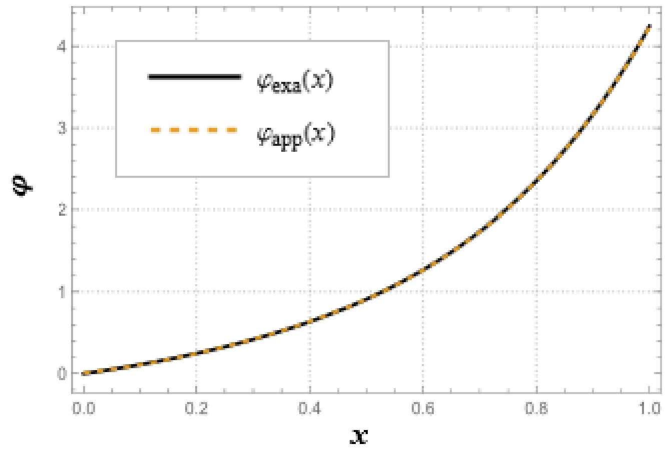

Figure 3.3(a): Exact and the approximate solutions of equation (3.11)

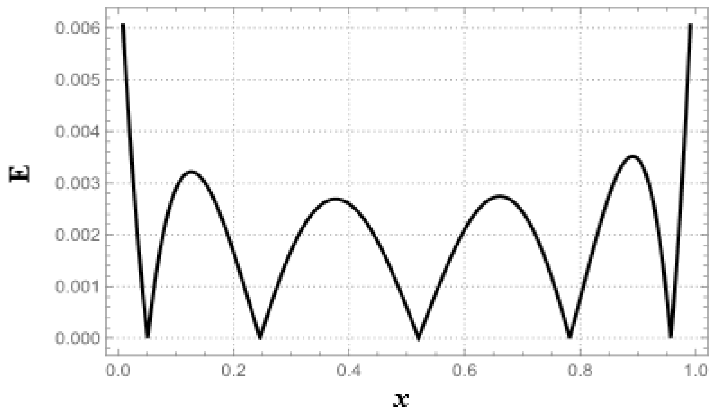

Figure 3.3(b): Absolute error for $n = 4$ of equation (3.11)

3.3 Bernstein polynomials method for solving linear Fredholm integral equation

We consider the linear Fredholm integral equation of the second kind given by

$$\mu\, \varphi(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t)dt \tag{3.12}$$

To solve this equation, we use the Bernstein polynomials method. Thus, we apply the same steps as the Bernstein polynomials method to the linear Volterra integral equation shown in section 3.2. The only difference is that the limits of integration in the linear Fredholm integral equation are constants.

3.3.1 Numerical Results and Discussion

To evaluate the effectiveness of the Bernstein polynomials method in solving the linear Fredholm integral equation, we present three numerical examples featuring the linear Fredholm integral equation with known exact solutions, the error estimation is provided to demonstrate the accuracy of the approximation.

Example 3.4. Find the solution of the following linear Fredholm integral equation of the second kind:

$$\varphi(x) = \frac{9}{10}x^2 + \frac{1}{2}\int_0^1 x^2t^2\varphi(t)dt \tag{3.13}$$

We have the exact solution (Yalcinbas et al., 2010) is:

$$\varphi_{exa}(x) = x^2.$$

Using Bernstein polynomials to solve equation (3.13) with $n = 5$, we have the following:

$$\varphi(x) = \frac{9}{10}x^2 + \frac{1}{2}\int_0^1 x^2t^2\varphi(t)dt$$

To determine the approximate solution of equation (3.13), we get

$$\varphi_{app}(x) = \sum_{i=0}^{5} a_i\, B_{i,5}(x) \tag{3.14}$$

Substituting equation (3.14) into equation (3.13), we obtain

$$\sum_{i=0}^{5} a_i\, B_{i,5}(x) = \frac{9}{10}x^2 + \frac{1}{2}\int_0^1 x^2t^2 \sum_{i=0}^{5} a_i\, B_{i,5}(t)dt$$

$$\sum_{i=0}^{5} a_i\, B_{i,5}(x) - \frac{1}{2}\int_0^1 x^2t^2 \sum_{i=0}^{5} a_i\, B_{i,5}(t)dt = \frac{9}{10}x^2$$

$$\sum_{i=0}^{5} a_i \left[B_{i,5}(x) - \frac{1}{2}\int_0^1 (x^2t^2)B_{i,5}(t)dt\right] = \frac{9}{10}x^2$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,\dots,5)$ and integrating both sides concerning x from -1 to 1, we have

$$\sum_{i=0}^{5} a_i \int_0^1 \left[B_{i,5}(x) - \frac{1}{2}\int_0^1 (x^2t^2)B_{i,5}(t)dt\right] B_{j,5}(x)dx = \int_0^1 \frac{9}{10}x^2B_{j,5}(x)dx \tag{3.15}$$

Now, we suppose that:

$$Q_{ij} = \int_0^1 \left[B_{i,5}(x) - \frac{1}{2}\int_0^1 (x^2t^2)B_{i,5}(t)dt\right] B_{j,5}(x)dx\,,\ \ i,j = 0,1,\dots,5$$

And

$$f_j = \int_0^1 \frac{9}{10}x^2B_{j,5}(x)dx\,, j = 0,1,\dots,5$$

Then equations (3.15) can be rewritten as

$$\sum_{i=0}^{n} a_i Q_{ij} = f_j \tag{3.16}$$

We can calculate integrals Q_{ij} and f_j, and substitute them back into equation (3.16), we get system of linear equations

$$\frac{56437}{620928}a_0 + \frac{9397}{206976}a_1 + \frac{6239}{310464}a_2 + \frac{2297}{310464}a_3 + \frac{131}{68992}a_4 - \frac{1}{88704}a_5 = 0.00535714$$
$$\frac{9397}{206976}a_0 + \frac{31261}{620928}a_1 + \frac{3887}{103488}a_2 + \frac{2185}{103488}a_3 + \frac{5105}{620928}a_4 + \frac{31}{29568}a_5 = 0.0160714$$
$$\frac{6239}{310464}a_0 + \frac{3887}{103488}a_1 + \frac{2207}{51744}a_2 + \frac{5435}{155232}a_3 + \frac{2075}{103488}a_4 + \frac{79}{14784}a_5 = 0.0321429$$
$$\frac{2297}{310464}a_0 + \frac{2185}{103488}a_1 + \frac{5435}{155232}a_2 + \frac{6445}{155232}a_3 + \frac{1215}{34496}a_4 + \frac{731}{44352}a_5 = 0.0535714$$
$$\frac{131}{68992}a_0 + \frac{5105}{620928}a_1 + \frac{2075}{103488}a_2 + \frac{1215}{34496}a_3 + \frac{28885}{620928}a_4 + \frac{393}{9856}a_5 = 0.0803571$$
$$\frac{-1}{88704}a_0 + \frac{31}{29568}a_1 + \frac{79}{14784}a_2 + \frac{731}{44352}a_3 + \frac{393}{9856}a_4 + \frac{117}{1408}a_5 = 0.1125$$

Therefore, we solve the above linear system of equation we get
$a_0 = -4.0462 \times 10^{-14}$, $a_1 = 2.0164 \times 10^{-13}$, $a_2 = 0.1$, $a_3 = 0.3$, $a_4 = 0.6$, $a_5 = 1$

Then, we substitute these values back into equation (3.14) to get the approximate solution of equation (3.13) for $n = 5$ as follows

$$\varphi_{app}(x) = -4.046 \times 10^{-14} + 1.211 \times 10^{-12}x + 0.1x^2 + 2.331 \times 10^{-11}x^3 - 2.656 \times 10^{-11}x^4 + 1.074 \times 10^{-11}x^5$$

Table 3.4 and Figure 3.4(a) present the exact and the approximate solutions calculated in some values of x over the interval $[0,1]$. The proposed method is applied for $n = 5$. The absolute error is also calculated for each value and has an order of approximately 10^{-15}. Figure 3.4(b) illustrates the behaviour of the absolute error for $n = 5$. The obtained results confirm that the used Bernstein polynomials mothed approximates the solution in a good way throughout the interval.

Table 3.4. Numerical results and absolute errors of equation (3.13).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	0	$-4.046163620 * 10^{-14}$	$4.046163620322 * 10^{-14}$
0.1	0.010000000000000	0.010000000000015	$1.53314860806 * 10^{-14}$

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0.2	0.040000000000000	0.040000000000004	$4.98212582309 * 10^{-15}$
0.3000000004	0.090000000000000	0.089999999999988	$1.117161918529 * 10^{-14}$
0.4	0.160000000000000	0.159999999999989	$1.088018564132 * 10^{-14}$
0.5	0.25	0.250000000000003	$3.10862446895 * 10^{-15}$
0.6	0.36	0.360000000000015	$1.576516694967 * 10^{-14}$
0.7	0.489999999999999	0.490000000000012	$1.260103132949 * 10^{-14}$
0.7999999999	0.639999999999999	0.639999999999992	$7.21644966006 * 10^{-15}$
0.8999999999	0.809999999999999	0.809999999999981	$1.809663530139 * 10^{-14}$
0.9999999999	0.999999999999999	1.000000000000044	$4.485301019485 * 10^{-14}$

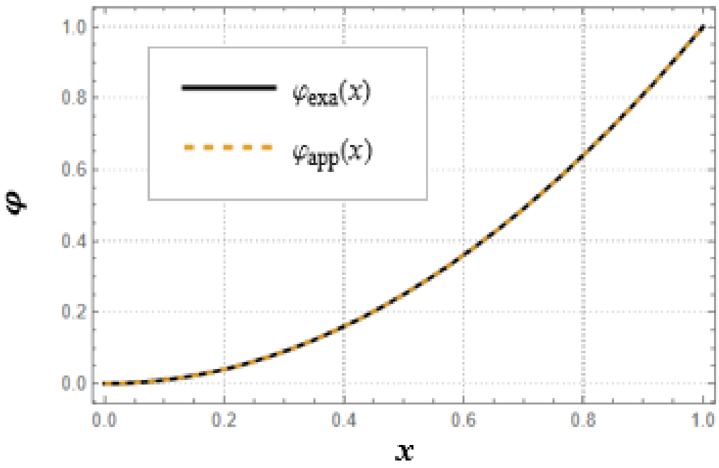

Figure 3.4(a): The exact and the approximate solutions of equation (3.13)

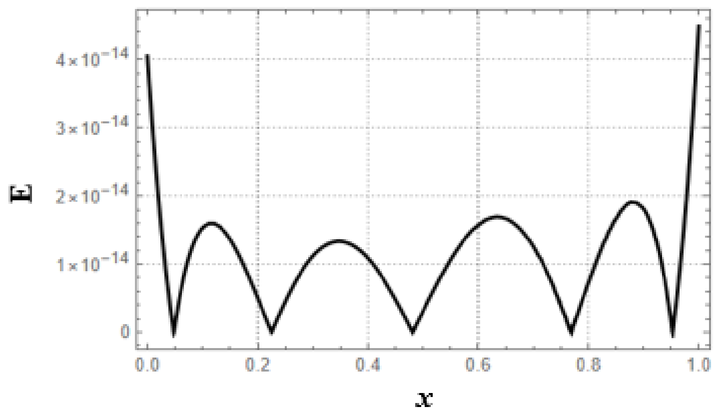

Figure 3.4(b): The absolute error of equation (3.13)

Example 3.5. We consider the linear Fredholm integral equation of the second kind:

$$\varphi(x) = \cos(x) - \sin(x) + 2x - \frac{\pi}{2}x + \int_0^{\pi/2} xt\varphi(t)dt \tag{3.17}$$

where the exact solution(Wazwaz, 2015) is:

$$\varphi(x) = \cos(x) - \sin(x).$$

We Apply the algorithm of the Bernstein polynomials method with degree $n = 3$, to obtain the linear system

$$\begin{aligned} 0.209a_0 + 0.082a_1 - 0.001a_2 - 0.049a_3 &= 0.298 \\ 0.082a_0 + 0.074a_1 + 0.009a_2 - 0.077a_3 &= 9.287 \\ -0.001a_0 + 0.009a_1 - 0.002a_2 - 0.070a_3 &= -9.290 \\ -0.049a_0 - 0.077a_1 - 0.070a_2 - 0.019a_3 &= -0.034 \end{aligned}$$

we solve the above linear system of equation, we get

$a_0 = 0.999$, $a_1 = 0.480$, $a_2 = -0.480$, $a_3 = -0.999$

Then, we get the approximate solution of equation (3.17) for $n = 3$ as follows

$$\varphi_{app}(x) = 0.999 - 0.992x - 0.537x^2 + 0.228x^3$$

Table 3.5 displays the exact and the approximate solutions calculated in some values of x over $[0,1]$. The proposed method is applied with $n = 3$. The absolute error is also calculated for each value and has an order of approximately 10^{-4}. It should be noted that, increasing n leads to good convergence of the approximate solution. Figure 3.5(b) illustrates the behaviour of the absolute error for $n = 7$. The obtained results in Table 3.5 and Figure 3.5(a) confirm that the Bernstein polynomials mothed is very efficient technique.

Table 3.5. Exact and Absolute error of equation (3.17).

x	Exact solution $\varphi_{exa}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$	
		$n = 3$	$n = 7$
0	1	$4.34839682812 * 10^{-4}$	$1.889890684170 * 10^{-7}$
0.1	0.895170748631197	$8.35814570986 * 10^{-5}$	$7.788218834292 * 10^{-7}$
0.2	0.781397247046180	$1.80812479275 * 10^{-4}$	$3.046555674401 * 10^{-7}$
0.3000000004	0.659816282464266	$8.64894835893 * 10^{-4}$	$5.940868181841 * 10^{-9}$
0.4	0.531642651694234	$477610195521 * 10^{-5}$	$3.881058807664 * 10^{-7}$
0.5	0.398157023286169	$1.36184261045 * 10^{-4}$	$1.931882118721 * 10^{-8}$
0.6	0.260693141514642	$1.46101196273 * 10^{-4}$	$5.466510860641 * 10^{-8}$

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0.7	0.120624500046797	$8.45821731918 * 10^{-5}$	$4.049243607917 * 10^{-7}$
0.7999999999	0.020649381552357	$1.52765623779 * 10^{-5}$	$7.701693768846 * 10^{-8}$
0.8999999999	0.161716941356818	$1.08336403621 * 10^{-4}$	$4.953591264955 * 10^{-7}$
0.9999999999	0.301168678939756	$1.51520242896 * 10^{-4}$	$4.912679094637 * 10^{-8}$

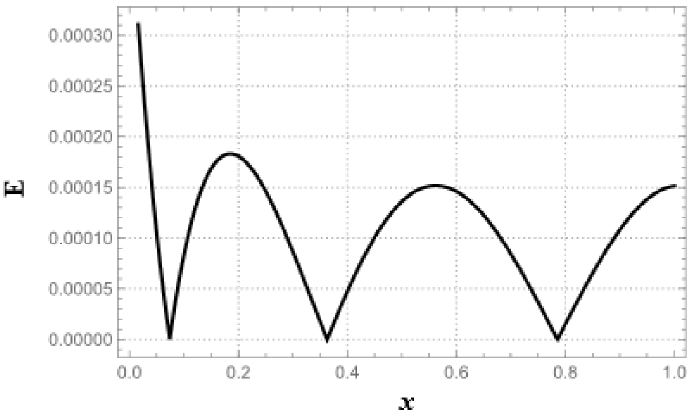

Figure 3.5(a): The absolute error for $n = 3$ of equation (3.17)

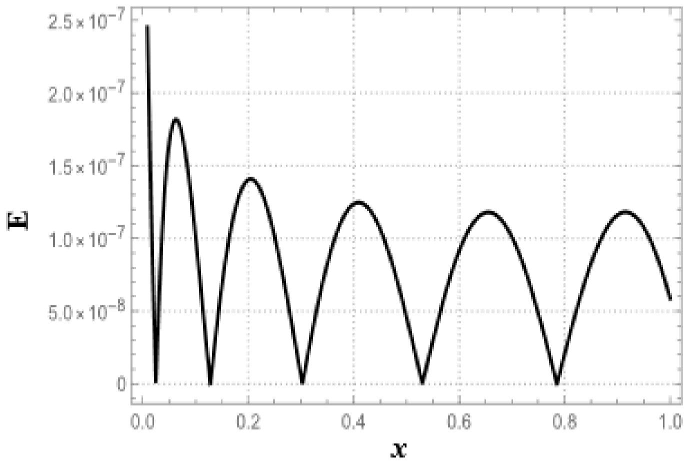

Figure 3.5(b): The absolute error for $n = 3$ of equation (3.17)

Example 3.6. Solve the following linear Fredholm integral equation of the second kind:

$$\varphi(x) = \frac{3}{2}e^{x} - \frac{1}{2}e^{x+2} + \int_{0}^{1} e^{x+t}\varphi(t)dt \tag{3.18}$$

where the exact solution(Jaber & Alrammahi, 2020) as,

$$\varphi(x) = e^{x}.$$

We apply the algorithm of the proposed method with degree $n = 4$ for Bernstein polynomials, to get the approximate solution of equation (3.18) as

$$\varphi_{app}(x) = 1 + 0.998x + 0.511x^{2} + 0.1397 + 0.069x^{4}.$$

Table 3.6 and Figure 3.6(a) present some values of the exact and the approximate solutions calculated at some different values on $[0,1]$. The Bernstein polynomials method is applied with $n = 4$. The absolute error is calculated for each value and has an order of approximately 10^{-4}all over $[0,1]$. Jaber & Alrammahi (Jaber & Alrammahi, 2020) solved equation (3.18) by using Spline Technique with $n = 10$, where the absolute error has an order of approximately 10^{-5} . The comparison of Absolute error of the proposed method and those in (Jaber & Alrammahi, 2020), showing the Bernstein polynomials method having a higher accuracy than in (Jaber & Alrammahi, 2020) with $n = 4$.

Table 3.6. Numerical results and absolute errors of equation (3.18).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	1	1.000052641788067	$5.264178806752 * 10^{-4}$
0.1	1.105170918075647	1.105149881666177	$2.103640947015 * 10^{-4}$
0.2	1.221402758160169	1.221393956029975	$8.802130194052 * 10^{-6}$
0.3000000004	1.349858807576003	1.349872974014913	$1.416643891016 * 10^{-4}$
0.4	1.491824697641270	1.491841799491063	$1.710184979319 * 10^{-4}$
0.5	1.648721270700128	1.648722051063122	$7.803629937797 * 10^{-7}$
0.6	1.822118800390508	1.822102102070406	$1.669832010287 * 10^{-4}$
0.7	2.013752707470476	2.013737080586854	$1.562688362222 * 10^{-4}$
0.7999999999	2.225540928492467	2.225548869421029	$7.940928561822 * 10^{-6}$
0.8999999999	2.459603111156949	2.459626106116113	$2.299495916435 * 10^{-4}$
0.9999999999	2.718281828459045	2.718224182949912	$5.764550913278 * 10^{-4}$

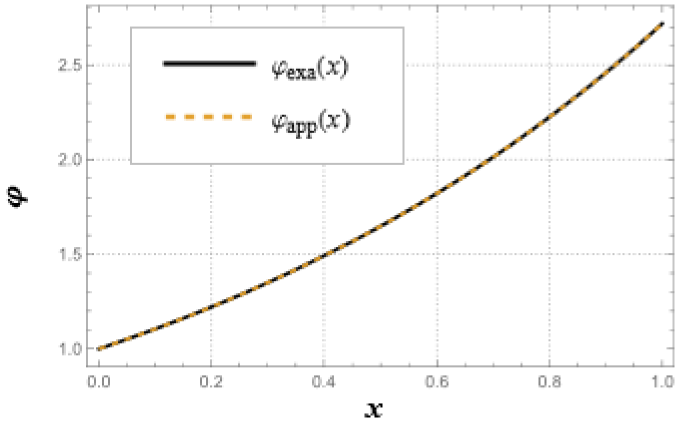

Figure 3.6(a): The exact and the approximate solutions of equation (3.18)

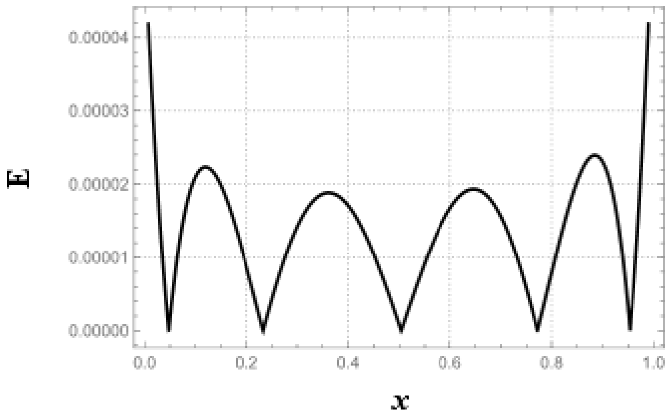

Figure 3.6(b): The absolute error of equation (3.18)

3.4 Bernstein polynomials method for solving linear Volterra-Fredholm integral equation

We consider the linear Volterra-Fredholm integral equation given by

$$\mu\,\varphi(x) = f(x) + \lambda_1 \int_a^b k_1(x,t)\varphi(t)dt + \lambda_2 \int_a^x k_2(x,t)\varphi(t)dt \tag{3.19}$$

In this section, the equation (3.19) will be solved by using Bernstein polynomials. We assume that the approximate solution $\varphi(x)$ of equation (3.19) is a function approximate of Bernstein polynomials defined in equation (2.19) as

$$\varphi(x) \simeq \sum_{i=0}^{n} a_i\, B_{i,n}(x) \tag{3.20}$$

Where $B_{i,n}(x)$ are the Bernstein polynomials of degree n are defined in equation (2.1) and a_i are coefficients to be determined. Substituting equation (3.20) into equation (3.19), we obtain

$$\mu\left[\sum_{i=0}^{n} a_i B_{i,n}(x)\right] = f(x) + \lambda_1 \int_a^b \left[k_1(x,t)\sum_{i=0}^{n} a_i B_{i,n}(t)\right] dt$$

$$+\lambda_2 \int_a^x \left[k_2(x,t)\sum_{i=0}^{n} a_i B_{i,n}(t)\right] dt\,, a < x < b.$$

$$\mu\left[\sum_{i=0}^{n} a_i B_{i,n}(x)\right] - \lambda_1 \int_a^b \left[k_1(x,t)\sum_{i=0}^{n} a_i B_{i,n}(t)\right] dt$$

$$-\lambda_2 \int_a^x \left[k_2(x,t)\sum_{i=0}^{n} a_i B_{i,n}(t)\right] dt = f(x)$$

$$\sum_{i=0}^{n} a_i \left[\mu\, B_{i,n}(x) - \lambda_1 \int_a^b k_1(x,t) B_{i,n}(t) dt - \lambda_2 \int_a^x k_2(x,t) B_{i,n}(t) dt\right] = f(x)$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,\dots,n)$ and integrating both sides concerning x from a to b, we have

$$\sum_{i=0}^{n} a_i \int_a^b \left[\mu\, B_{i,n}(x) - \lambda_1 \int_a^b k_1(x,t) B_{i,n}(t) dt \right.$$
$$\left. - \lambda_2 \int_a^x k_2(x,t) B_{i,n}(t) dt\right] B_{j,n}(x) dx = \int_a^b f(x) B_{j,n}(x) dx \tag{3.21}$$

Since above equation, there are four integrals, the inner integrand of the left side is a function of x and t and is integrated concerning to t from a to b. As a result, the outer integrand becomes a function of x only and integration concerning to x yields a constant. We suppose that:

$$Q_{ij} = \int_a^b \left[\mu\, B_{i,n}(x) - \lambda_1 \int_a^b k_1(x,t) B_{i,n}(t) dt - \lambda_2 \int_a^x k_2(x,t) B_{i,n}(t) dt\right] B_{j,n}(x) dx,$$

and

$$f_j = \int_a^b f(x) B_{j,n}(x) dx$$

Then equation (3.21) can be rewritten as,

$$\sum_{i=0}^{n} a_i Q_{ij} = f_j \tag{3.22}$$

Thus, for each $j = 0,1,2,\dots,n$ we have a linear equation with $(n+1)$ unknown coefficients $a_i, i = 0,1,2,\dots,n$. Therefore, the linear system (3.22) can be solved by any standard method to produce a_i and substituting these values back in equation (3.20), we get the approximate solution $\varphi(x)$ of the linear Volterra-Fredholm integral equation.

The Algorithm of the method.

The approximate solution of the linear Volterra-Fredholm integral equation, by using the Bernstein polynomials method is coded using the Wolfram Mathematica version 14.1, as shown in Appendix A.

Step 1: Input the constants a, b, c, n, λ_1, λ_2.

Step 2: Input the functions μ, $f(x)$, $k_1(x,t), k_2(x,t)$, $B_{i,n}(t)$.

Step 3: Calculate the integrations Q_{ij}, f_j.

Step 4: Calculate the unknown coefficients a_i.

Step 5: Calculate the approximate solution $\varphi(x)$.

Step 6: Compute the absolute error E.

3.4.1 Convergence and uniqueness of the numerical solution of linear Volterra-Fredholm integral equation

Here we show that the approximate (numerical) solution of the Volterra-Fredholm integral equation obtained from the Bernstein polynomials method converges to the exact solution.

Theorem 3.1. (Uniqueness Theorem) (Mustapha, 2022) Let $\varphi(x)$ given by (3.20) be a solution of the equation (3.19). Then the solution of the Volterra-Fredholm integral equation by the Bernstein polynomials method is unique.

Proof. To prove the uniqueness, Let $\varphi_1(x)$ and $\varphi_2(x)$ be two different solutions for equation (3.19), so

$$|\varphi_1(x) - \varphi_2(x)| = \left|f(x) + \lambda_1 \int_a^b k_1(x,t)\varphi_1(t)dt + \lambda_2 \int_a^x k_2(x,t)\varphi_1(t)dt \right.$$

$$\left. - \left[f(x) + \lambda_1 \int_a^b k_1(x,t)\varphi_2(t)dt + \lambda_2 \int_a^x k_2(x,t)\varphi_2(t)dt\right]\right|$$

$$= \left|\lambda_1 \int_a^b k_1(x,t)\varphi_1(t)dt + \lambda_2 \int_a^x k_2(x,t)\varphi_1(t)dt \right.$$

$$\left. -\lambda_1 \int_a^b k_1(x,t)\varphi_2(t)dt - \lambda_2 \int_a^x k_2(x,t)\varphi_2(t)dt\right|$$

$$= \left|\lambda_1 \int_a^b k_1(x,t)[\varphi_1(t) - \varphi_2(t)]dt \right.$$

$$\left. +\lambda_2 \int_a^x k_2(x,t)[\varphi_1(t) - \varphi_2(t)]dt\right|$$

$$\leq \left|\lambda_1 \int_a^b k_1(x,t)[\varphi_1(t)-\varphi_2(t)]dt\right|$$

$$+\left|\lambda_2 \int_a^x k_2(x,t)[\varphi_1(t)-\varphi_2(t)]dt\right|.$$

Now we use Cauchy -Schwarz inequality given by equation (1.15) to obtain

$$\leq |\lambda_1|\left|\int_a^b k_1(x,t)[\varphi_1(t)-\varphi_2(t)]dt\right|$$

$$+|\lambda_2|\left|\int_a^x k_2(x,t)[\varphi_1(t)-\varphi_2(t)]dt\right|$$

$$\leq |\lambda_1|M_1 \int_a^b |\varphi_1(t)-\varphi_2(t)|\,dt + |\lambda_2|M_2 \int_a^x |\varphi_1(t)-\varphi_2(t)|dt,$$

where $|k_1(x,t)| \leq M_1$ and $|k_2(x,t)| \leq M_2$. Since $a \leq x \leq b$,we have

$$|\varphi_1(x)-\varphi_2(x)| \leq (|\lambda_1|+|\lambda_2|)M \int_a^b |\varphi_1(t)-\varphi_2(t)|$$

$$\leq (|\lambda_1|+|\lambda_2|)M\beta|\varphi_1(t)-\varphi_2(t)|$$

$$= \alpha|\varphi_1(t)-\varphi_2(t)|,$$

where $M = max\{M_1, M_2\}$, $\alpha = (|\lambda_1|+|\lambda_2|)M\beta$, $\beta = (b-a)$. Then

$$|\varphi_1(t)-\varphi_2(t)| \leq \alpha|\varphi_1(t)-\varphi_2(t)|,$$

$$(1-\alpha)|\varphi_1(t)-\varphi_2(t)| \leq 0$$

Since $< \alpha < 1$, then $|\varphi_1(t)-\varphi_2(t)| = 0$,then

$$\varphi_1(t) = \varphi_2(t)$$

Hence the solution of the Volterra-Fredholm integral equation is unique.

Theorem 3.2. (Convergence Theorem) (SALIM et al., 2022) Let $f(x)$ be a function defined on $[a,b]$. Let $k_1(x,t)$ and $k_2(x,t)$ be in $[a,b]\times[0,t_1]$. Assume that the known functions $f(x)$, $k_1(x,t)$ and $k_2(x,t)$ are smooth continuous and arbitrary differentiable functions in the equation (3.19). Let $\varphi_{exa}(t)$ be the exact solution of equation (3.19) and $\varphi_{app}(t)$ is the approximate solution obtained by the Bernstein polynomials method. Then

$$\lim_{n\to\infty}\left\|\varphi_{exa}(t)-\varphi_{app}(t)\right\| = 0$$

Proof. We start the proof the convergence from definition of the norms as

$$\left\|\varphi_{exa}(t)-\varphi_{app}(t)\right\| = \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\left|\varphi_{exa}(t)-\varphi_{app}(t)\right|$$

$$= \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\left|f(x)+\lambda_1\int_a^b k_1(x,t)\varphi_{exa}(t)dt+\lambda_2\int_a^x k_2(x,t)\varphi_{exa}(t)dt\right.$$

$$-\left[f(x)+\lambda_1\int_a^b k_1(x,t)\varphi_{app}(t)dt+\lambda_2\int_a^x k_2(x,t)\varphi_{app}(t)dt\right]\Bigg\|$$

$$=\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\left|\lambda_1\int_a^b k_1(x,t)\varphi_{exa}(t)dt+\lambda_2\int_a^x k_2(x,t)\varphi_{exa}(t)dt\right.$$

$$\left.-\lambda_1\int_a^b k_1(x,t)\varphi_{app}(t)dt-\lambda_2\int_a^x k_2(x,t)\varphi_{app}(t)dt\right|$$

$$=\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\left|\lambda_1\int_a^b k_1(x,t)\left[\varphi_{exa}(t)-\varphi_{app}(t)\right]dt\right.$$

$$\left.+\lambda_2\int_a^x k_2(x,t)\left[\varphi_{exa}(t)-\varphi_{app}(t)\right]dt\right|$$

$$\leq\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\left|\lambda_1\int_a^b k_1(x,t)\left[\varphi_{exa}(t)-\varphi_{app}(t)\right]dt\right|$$

$$+\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\left|\lambda_2\int_a^x k_2(x,t)\left[\varphi_{exa}(t)-\varphi_{app}(t)\right]dt\right|.$$

Now we use Cauchy -Schwarz inequality given by equation (1.15), we obtain

$$\leq|\lambda_1|\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\int_a^b\left|k_1(x,t)\left[\varphi_{exa}(t)-\varphi_{app}(t)\right]\right|dt$$

$$+|\lambda_2|\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\int_a^b\left|k_2(x,t)\left[\varphi_{exa}(t)-\varphi_{app}(t)\right]\right|dt$$

We suppose $L_1=\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}|k_1(x,t)|$ and $L_2=\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}|k_2(x,t)|$. Then we get

$$\leq|\lambda_1|L_1\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\int_a^b\left|\varphi_{exa}(t)-\varphi_{app}(t)\right|dt$$

$$+|\lambda_2|L_2\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\int_a^x\left|\varphi_{exa}(t)-\varphi_{app}(t)\right|dt$$

Since $a\leq x\leq b$, and $L=max\{L_1,L_2\}$, we have

$$\left\|\varphi_{exa}(t)-\varphi_{app}(t)\right\|\leq[|\lambda_1|+|\lambda_2|]L\max_{\substack{x\in[a,b]\\ t\in[0,t_1]}}\int_a^b\left|\varphi_{exa}(t)-\varphi_{app}(t)\right|$$

$$\leq[|\lambda_1|+|\lambda_2|]L(b-a)\left\|\varphi_{exa}(t)-\varphi_{app}(t)\right\|_\infty$$

We suppose $\alpha=[|\lambda_1|+|\lambda_2|]L\beta$, where $\beta=(b-a)$.

$$(1-\alpha)\left\|\varphi_{exa}(t)-\varphi_{app}(t)\right\|_\infty=0$$

Therefore if $0 \leq \alpha \leq 1$ and $n \to \infty$,we have

$$\lim_{n\to\infty} \left\| \varphi_{exa}(t) - \varphi_{app}(t) \right\| = 0$$

3.4.2 Numerical Results and Discussion

In this section, various examples are presented demonstrate the efficiency of the method described we will compare the exact solution $\varphi_{exa}(x)$ with the approximate solution $\varphi_{app}(x)$, and we calculate the absolute error E.

Example 3.7. Find the solution of the linear Volterra-Fredholm integral equation of the second kind:

$$\varphi(x) = -\frac{2}{5}x^7 - \frac{5}{4}x^4 + x^3 - \frac{59}{20}x + 1 + \int_0^1 x(t+1)\varphi(t)dt + \int_0^x (2x^2t+1)\varphi(t)dt \tag{3.23}$$

where the exact solution (Elmaci & Baykus Savasaneril, 2022) is given as,

$$\varphi(x) = x^3 + 1.$$

To determine the approximate solution of equation (3.23), we get

$$\varphi_{app}(x) = \sum_{i=0}^{4} a_i\, B_{i,4}(x) \tag{3.24}$$

Substituting equation (3.24) into equation (3.23), leads to

$$\sum_{i=0}^{4} a_i\, B_{i,4}(x) = -\frac{2}{5}x^7 - \frac{5}{4}x^4 + x^3 - \frac{59}{20}x + 1 + \int_0^1 x(t+1)\sum_{i=0}^{4} a_i\, B_{i,4}(t)dt + \int_0^x (2x^2t+1)\sum_{i=0}^{4} a_i\, B_{i,4}(t)dt$$

$$\sum_{i=0}^{4} a_i\, B_{i,4}(x) - \int_0^1 x(t+1)\sum_{i=0}^{4} a_i\, B_{i,4}(t)dt - \int_0^x (2x^2t+1)\sum_{i=0}^{4} a_i\, B_{i,4}(t)dt = -\frac{2}{5}x^7 - \frac{5}{4}x^4 + x^3 - \frac{59}{20}x + 1$$

$$\sum_{i=0}^{4} a_i \left[B_{i,4}(x) - \int_0^1 x(t+1)B_{i,4}(t)dt - \int_0^x (2x^2t+1)B_{i,4}(t)dt \right] = -\frac{2}{5}x^7 - \frac{5}{4}x^4 + x^3 - \frac{59}{20}x + 1$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1, \dots, 4)$ and integrating both sides concerning x from 0 to 1, to obtain

$$\sum_{i=0}^{4} a_i \int_0^1 \left[B_{i,4}(x) - \int_0^1 x(t+1)B_{i,4}(t)dt - \int_0^x (2x^2t+1)B_{i,4}(t)dt \right] B_{j,4}(x)dx \tag{3.25}$$
$$= \int_0^1 \left[-\frac{2}{5}x^7 - \frac{5}{4}x^4 + x^3 - \frac{59}{20}x + 1 \right] B_{j,4}(x)dx$$

Now, we suppose that:

$$Q_{ij} = \int_0^1 \left[B_{i,4}(x) - \int_0^1 x(t+1)B_{i,4}(t)dt - \int_0^x (2x^2t+1)B_{i,4}(t)dt \right] B_{j,4}(x)dx, \quad i,j = 0,1, \dots, 4$$

And

$$f_j = \int_0^1 \left[-\frac{2}{5}x^7 - \frac{5}{4}x^4 + x^3 - \frac{59}{20}x + 1 \right] B_{j,4}(x)dx$$

Then equations (3.25) can be rewritten as

$$\sum_{i=0}^{4} a_i Q_{ij} = f_j \tag{3.26}$$

We can calculate integrals Q_{ij} and f_j, and substitute them back into equation (3.26), we get system of linear equations

$$\frac{74699}{900900}a_0 + \frac{16781}{450450}a_1 + \frac{454}{45045}a_2 - \frac{5843}{1351350}a_3 - \frac{7328}{675675}a_4 = \frac{14297}{138600}$$
$$\frac{2}{273}a_0 + \frac{421}{18018}a_1 + \frac{449}{30030}a_2 - \frac{19}{8190}a_3 - \frac{323}{18018}a_4 = \frac{191}{27720}$$
$$\frac{-6577}{12870}a_0 + \frac{6577}{450450}a_1 + \frac{43}{100100}a_2 - \frac{106}{75075}a_3 - \frac{927}{50050}a_4 = \frac{-2141}{23100}$$
$$\frac{-13207}{193050}a_0 - \frac{15577}{270270}a_1 - \frac{1843}{50050}a_2 - \frac{6869}{450450}a_3 - \frac{7}{650}a_4 = \frac{-28201}{138600}$$
$$\frac{-39019}{450450}a_0 - \frac{3031}{32175}a_1 - \frac{2238}{25025}a_2 - \frac{181}{2925}a_3 + \frac{17}{3900}a_4 = \frac{-61}{180}$$

We solve the linear system of equation, we get

$a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1.25$, $a_4 = 2$

Then, we substitute these values back into equation (3.24) to get the approximate solution of equation (3.23) for $n = 4$ as follows

$$\varphi_{app}(x) = 0.999 + 4.884 \times 10^{-15}x - 3.019 \times 10^{-14}x^2 + x^3 - 3.286 \times 10^{-14}x^4$$

Elmaci & Baykus Savasaneril (2022) solved equation (3.23) by the so-called Lucas polynomials method with $n = 3$, where the error functions of equation (3.23) has an order of $2.5313e^{-14}$. Our numerical results obtained by Bernstein polynomials for $n = 4$ tabulated in Table 3.7 are of order 10^{-16}. One could claim that the Bernstein polynomials perform better than the Lucas polynomials method.

Table 3.7. Exact and approximate solutions of equation (3.23).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	1	0.999999999999999	$7.771561172376 * 10^{-16}$
0.1	1.001	1.000999999999999	$4.4408920985 * 10^{-16}$
0.2	1.008	1.007999999999999	$6.661338147750 * 10^{-16}$
0.3000000004	1.027	1.026999999999999	$6.66133814775 * 10^{-16}$
0.4	1.064	1.063999999999999	$8.881784197001 * 10^{-16}$
0.5	1.125	1.124999999999999	$4.440892098500 * 10^{-16}$
0.6	1.216	1.215999999999999	$2.220446049250 * 10^{-16}$
0.7	1.343	1.343	0
0.7999999999	1.512	1.512000000000000	$2.220446049250 * 10^{-16}$
0.8999999999	1.728999999999999	1.729	$4.440892098506 * 10^{-16}$
0.9999999999	1.999999999999999	1.999999999999999	$4.440892098500 * 10^{-16}$

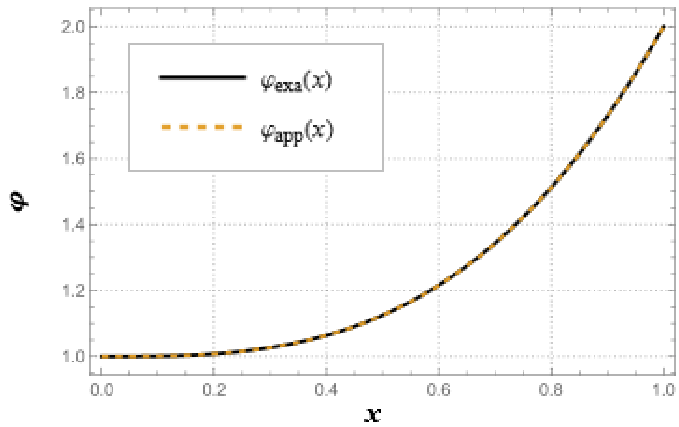

Figure 3.7(a): Exact and the approximate solutions of equation (3.23)

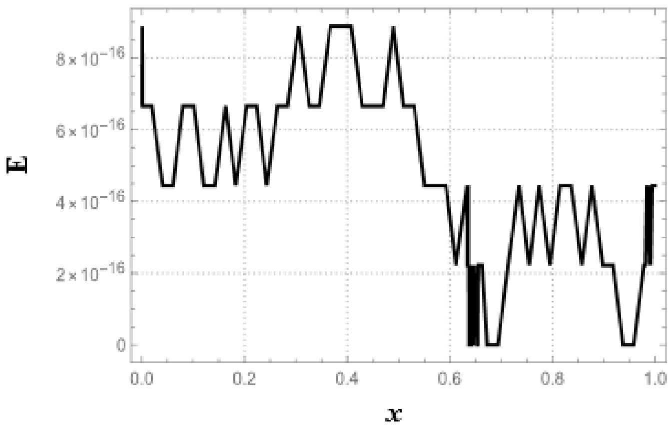

Figure 3.7(b): Absolute error for $n = 4$ of equation (3.23)

Example 3.8. Solve the linear Volterra-Fredholm integral equation:

$$\varphi(x) = 2\cos(x) - xcos(2) - 2xsin(2) + x - 1 + \int_0^2 xt\varphi(t)dt + \int_0^x (x-t)\varphi(t)dt \tag{3.27}$$

We have the exact solution (Geçmen & Çelik, 2021) is:

$$\varphi(x) = \cos(x).$$

By approximating the unknown $\varphi(x)$ by the Bernstein polynomials of degree $n = 6$, the integral equation (3.24) is reduced to the following linear system

$$\frac{5321}{38220}a_0 + \frac{57}{910}a_1 + \frac{39}{2156}a_2 - \frac{146}{21021}a_3 - \frac{1765}{84084}a_4 - \frac{3098}{105105}a_5 - \frac{44839}{1261260}a_6 = 0.226$$
$$\frac{269}{6370}a_0 + \frac{1763}{35035}a_1 + \frac{365}{13014}a_2 - \frac{128}{21021}a_3 - \frac{241}{7007}a_4 - \frac{5827}{105105}a_5 - \frac{7388}{105105}a_6 = 0.138$$
$$\frac{-1}{44}a_0 + \frac{79}{14014}a_1 + \frac{23}{2548}a_2 - \frac{215}{21021}a_3 - \frac{3529}{84084}a_4 - \frac{527}{7007}a_5 - \frac{8629}{84084}a_6 = 0.025$$
$$\frac{-1433}{21021}a_0 - \frac{986}{2102}a_1 - \frac{92}{3003}a_2 - \frac{1928}{63063}a_3 - \frac{1073}{21021}a_4 - \frac{1844}{21021}a_5 - \frac{2720}{21021}a_6 = -0.108$$
$$\frac{-8629}{84084}a_0 - \frac{670}{7007}a_1 - \frac{6961}{84084}a_2 - \frac{1502}{21021}a_3 - \frac{185}{2548}a_4 - \frac{193}{2002}a_5 - \frac{313}{2156}a_6 = -0.259$$
$$\frac{-13823}{105105}a_0 - \frac{14407}{10510}a_1 - \frac{956}{7007}a_2 - \frac{386}{3003}a_3 - \frac{1637}{14014}a_4 - \frac{3957}{35035}a_5 - \frac{901}{6370}a_6 = -0.420$$
$$\frac{-199279}{126126}a_0 - \frac{18113}{10510}a_1 - \frac{1549}{8408}a_2 - \frac{4007}{21021}a_3 - \frac{401}{2156}a_4 - \frac{1031}{6370}a_5 - \frac{577}{5460}a_6 = -0.587$$

We solve the above linear system of equation, we get

$a_0 = 1, a_1 = 0.999, a_2 = 0.866, a_3 = 0.599, a_4 = 0.245, a_5 = -0.112, a_6 = -0.4161.$

Then, we get the approximate solution of equation (3.24) when $n = 6$ as follows

$$\varphi_{app}(x) = 1 - 0.001x - 0.498x^2 - 0.003x^3 + 0.045x^4 - 0.0023x^5 - 0.001x^6$$

Merve and Ercan solved equation (3.24) by Hosoya polynomials method with $n = 8$, Maximum error analysis in (Geçmen & Çelik, 2021) is $1.6431e^{-12}$, This equation has been solved by Bernstein polynomials mothed with $n = 6$, the numerical results and absolute errors for equation (3.24) shown in Table 3.8, Figure 3.8(a) and Figure 3.8(b) confirm that the used Bernstein polynomials mothed approximates the solution a good manner throughout interval [0,2].

Table 3.8. The approximate and the exact solutions of equation (3.27).

x	Exact solution $\varphi_{exa}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$	
		$n = 6$	$n = 9$
0	1	$5.79332473749 * 10^{-6}$	$1.805035787505 * 10^{-9}$
0.2	0.980066577841241	$1.46061624739 * 10^{-6}$	$1.12748876909 * 10^{-9}$
0.4	0.921060994002885	$1.89131095862 * 10^{-6}$	$1.102057223256 * 10^{-9}$
0.6000000000	0.825335614909678	$6.47145304011 * 10^{-9}$	$1.875197774836 * 10^{-10}$
0.8	0.696706709347165	$1.76076696034 * 10^{-6}$	$1.852790254552 * 10^{-9}$
1	0.540302305868139	$7.08763456813 * 10^{-8}$	$1.122486104065 * 10^{-9}$
1.2	0.362357754476673	$1.78168018383 * 10^{-6}$	$1.726235931975 * 10^{-9}$
1.4	0.169967142900241	$1.44679878189 * 10^{-7}$	$2.622236067173 * 10^{-9}$
1.5999999999	0.029199522301288	$2.00229545487 * 10^{-6}$	$3.822594081548 * 10^{-10}$
1.7999999999	0.227202094693086	$1.47054117743 * 10^{-6}$	$1.672899402338 * 10^{-9}$
1.9999999999	0.416146836547142	$6.3129659913507 * 10^{-6}$	$1.215404710208 * 10^{-8}$

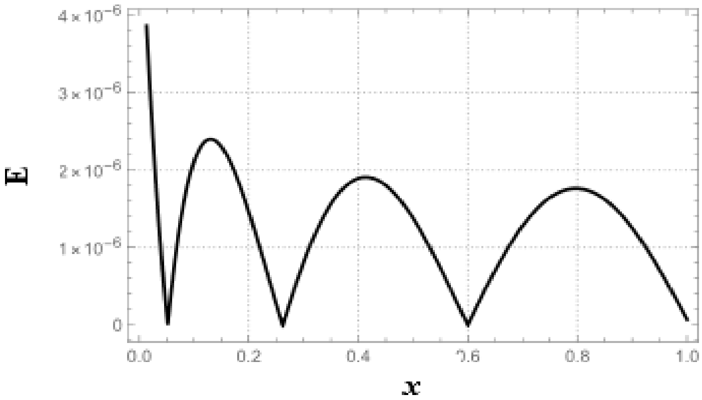

Figure 3.8(a): Absolute error for $n = 6$ of equation (3.27)

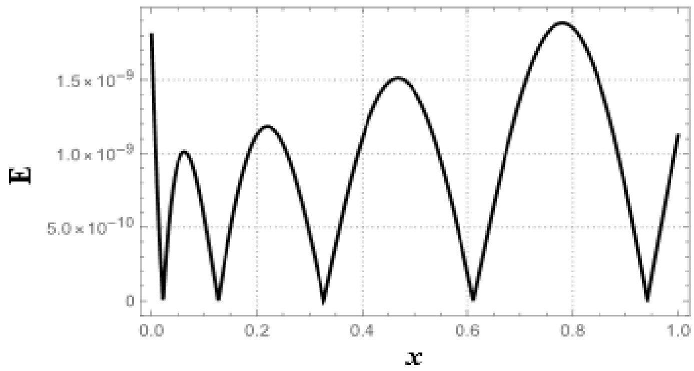

Figure 3.8(b): Absolute error for $n = 6$ of equation (3.27)

Example 3.9. We consider the following linear Volterra-Fredholm integral equation:

$$\varphi(x) = x - 2e^{x} + e^{-x} + 1 + \int_0^1 e^{(x+t)}\varphi(t)dt + \int_0^x te^{x}\varphi(t)dt \qquad (3.28)$$

where the exact solution (He et al., 2021) as,

$$\varphi_{ext}(x) = e^{-x}.$$

We apply the algorithm of the Bernstein polynomials method with degree $n = 4$, to obtain the approximate solution as,

$$\varphi_{app}(x) = 0.999 - 0.999x + 0.493x^2 - 0.153x^3 + 0.025x^4$$

Numerical results are listed in Table 3.9 illustrate the convergence of the approximate solution $\varphi_{app}(x)$ to the exact solution $\varphi_{ext}(x)$ at different values of x over $[0,1]$, The absolute error is calculated by Bernstein polynomials for each values and has an order of approximately 10^{-6} all over $[0,1]$. In (He et al., 2021) solved equation (3.25) by the improved block–pulse basis functions with $n = 16$, where the absolute error has an order of approximately 10^{-2}, Then the comparison of Absolute error of the proposed method and those in (He et al., 2021), showing the Bernstein polynomials method perfume much better than in (He et al., 2021) with only $n = 4$.

Table 3.9. Numerical results and absolute errors of equation (3.28).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	1	0.999978229988653	$2.177001134628 * 10^{-4}$
0.2	0.904837418035959	0.904846141303225	$8.723267266308 * 10^{-6}$
0.4	0.818730753077981	0.818733898662437	$3.145584455466 * 10^{-6}$
0.6000000000	0.740818220681717	0.740812348933822	$5.871747895236 * 10^{-6}$

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0.8	0.670320046035639	0.670313506473837	$6.539561802298 * 10^{-6}$
1	0.606530659712633	0.606530553127854	$1.065847786696 * 10^{-7}$
1.2	0.548811636094026	0.548817838230170	$6.202136143995 * 10^{-6}$
1.4	0.496585303791409	0.496590878603998	$5.574812589226 * 10^{-6}$
1.5999999999	0.449328964117221	0.449326358561474	$2.605555747603 * 10^{-6}$
1.7999999999	0.406569659740599	0.406562129903650	$7.529836948849 * 10^{-6}$
1.9999999999	0.367879441171442	0.367897211920502	$1.777074905984 * 10^{-4}$

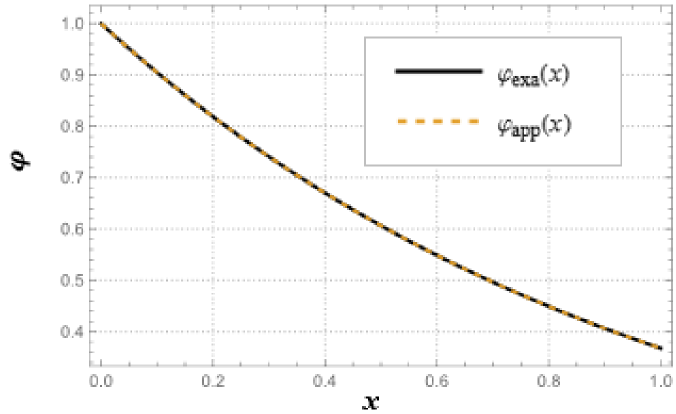

Figure 3.9(a): Exact and the approximate solutions of equation (4.28)

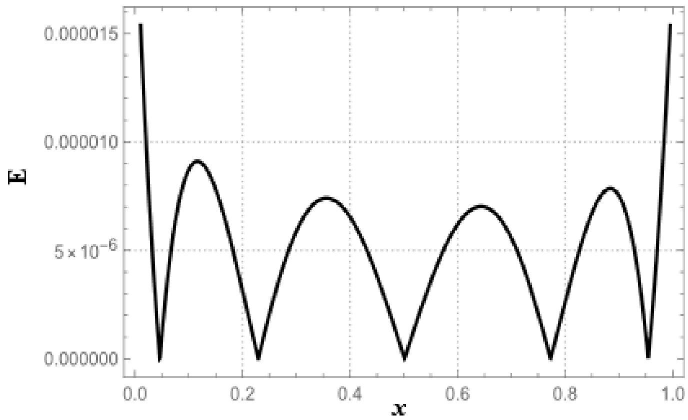

Figure 3.9(b): Absolute error for $n = 4$ of equation (4.28)

3.5 Bernstein polynomials method for solving linear mixed Volterra-Fredholm integral equation

We consider the linear mixed Volterra-Fredholm integral equation given by

$$\mu\,\varphi(x) = f(x) + \lambda \int_a^x \int_a^b k(y,t)\varphi(t)dt\,dy \tag{3.29}$$

In this section, we solve the linear mixed Volterra-Fredholm integral equation of the form (3.29) by using Bernstein polynomials. We assume that the approximate solution of equation (3.29) is an approximate function of Bernstein polynomials defined in equation (2.19) as

$$\varphi(x) \simeq \sum_{i=0}^{n} a_i\, B_{i,n}(x) \tag{3.30}$$

Where $B_{i,n}(x)$the Bernstein polynomials of degree n are defined in equation (2.1) and a_i are coefficients to be determined. Substituting (3.30) into (3.29), we obtain

$$\mu \sum_{i=0}^{n} a_i\, B_{i,n}(x) \;= f(x) + \lambda \int_a^x \int_a^b k(y,t) \sum_{i=0}^{n} a_i\, B_{i,n}(t)\;dt\,dy$$

$$\mu \sum_{i=0}^{n} a_i\, B_{i,n}(x) - \lambda \int_a^x \int_a^b k(y,t) \sum_{i=0}^{n} a_i\, B_{i,n}(t)\;dt\,dy \;= f(x)$$

$$\sum_{i=0}^{n} a_i \left[\mu\, B_{i,n}(x) - \lambda \int_a^x \int_a^b k(y,t) B_{i,n}(t)\;dt\,dy\right] = f(x)$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,\dots,n)$ and integrating both sides concerning x from a to b, we have

$$\sum_{i=0}^{n} a_i \int_a^b \left[\mu\, B_{i,n}(x) - \lambda \int_a^x \int_a^b k(y,t) B_{i,n}(t)\;dt\,dy\right] B_{j,n}(x)dx$$
$$= \int_a^b f(x) B_{j,n}(x)dx \tag{3.31}$$

Since above equation, there are four integrals, the inner integrand of the left side is a function of y and t and is integrated concerning to t from a to b. As a result, the outer integrand becomes a function of x only and integration concerning to x yields a constant. We suppose that:

$$Q_{ij} = \int_a^b \left[\mu\, B_{i,n}(x) - \lambda \int_a^x \int_a^b k(y,t) B_{i,n}(t)\;dt\,dy\right] B_{j,n}(x)dx$$

And

$$f_j = \int_a^b f(x) B_{j,n}(x) dx$$

Then equations (3.31) can be rewritten as

$$\sum_{i=0}^{n} a_i Q_{ij} = f_j \tag{3.32}$$

Thus, for each $j = 0,1,2, \dots, n$, we have a linear equation with $n+1$ unknown coefficients $a_i, i = 0,1,2, \dots, n$. Therefore, the linear system (3.32) can be solved by any standard method to produce a_i and substituting these values in (3.30), we get the approximate solution $\varphi(x)$ of the linear mixed Volterra-Fredholm integral equation.

3.5.1 Numerical Results and Discussion

We present numerical examples to demonstrate the accuracy of the proposed Bernstein polynomials method and to verify the error estimates. The computations associated with the examples are performed by the Wolfram Mathematica software version 14.1.

Example 3.10. Solve the linear mixed Volterra-Fredholm integral equation,

$$\varphi(x) = xe^x - \frac{1}{2}x^2 + \int_0^x \int_0^1 y\varphi(t) dt dy \tag{3.33}$$

where the exact solution (Hasan & Suleiman, 2018) is given as,

$$\varphi(x) = xe^x$$

We apply the algorithm of the proposed method with degree $n = 6$ for Bernstein polynomials, to obtain the following linear system,

$$\begin{bmatrix} \frac{3515}{45864} & \frac{1751}{45864} & \frac{8677}{504504} & \frac{3385}{504504} & \frac{1033}{504504} & \frac{151}{504504} & \frac{101}{504504} \\ \frac{575}{15288} & \frac{6913}{168168} & \frac{5149}{168168} & \frac{2993}{168168} & \frac{1327}{168168} & \frac{361}{168168} & \frac{15}{56056} \\ \frac{1327}{84084} & \frac{2503}{84084} & \frac{2797}{84084} & \frac{769}{28028} & \frac{358}{21021} & \frac{148}{21021} & \frac{53}{84084} \\ \frac{1049}{252252} & \frac{3989}{252252} & \frac{6635}{252252} & \frac{7685}{252252} & \frac{6635}{252252} & \frac{3989}{252252} & \frac{1049}{252252} \\ \frac{-323}{168168} & \frac{755}{168168} & \frac{2435}{168168} & \frac{1395}{56056} & \frac{5165}{168168} & \frac{4577}{168168} & \frac{2225}{168168} \\ \frac{-43}{8008} & \frac{-71}{24024} & \frac{67}{24024} & \frac{305}{24024} & \frac{613}{24024} & \frac{865}{24024} & \frac{71}{2184} \\ \frac{-283}{36036} & \frac{-265}{36036} & \frac{-101}{18018} & \frac{-17}{18018} & \frac{86}{9009} & \frac{25}{819} & \frac{113}{1638} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0.021 \\ 0.072 \\ 0.073 \\ 0.106 \\ 0.146 \\ 0.193 \\ 0.249 \end{bmatrix}$$

we solve the linear above system of equation, to get

$a_0 = 7.057 \times 10^{-7}, a_1 = 0.166 , a_2 = 0.400 , a_3 = 0.724 , a_4 = 1.177 , a_5 = 1.812$ and $a_6 = 2.718$

Then, we get the approximate solution of equation (3.33) when for $n = 6$ as follows

$\varphi_{app}(x) = 7.057 \times 10^{-7} + 0.999x + x^2 + 0.497x^3 + 0.174x^4 + 0.031x^5 + 0.015x^6$

Hasan & Suleiman, (Hasan & Suleiman, 2018) solved equation (3.33) by technique is based on a polynomials with $n = 10$ and solved by the Simplex method, where the absolute error was of order of approximately 10^{-7}. We solved equation (3.33) by Bernstein polynomials method with $n = 6$, we obtain the numerical results shown in table 3.10 and Figure 3.10(a), also the absolute error is calculated for each value and has an order of approximately 10^{-7}all over $[0,1]$. Hence it justifies the ability and efficiency of using the Bernstein polynomials.

Table 3.10. Exact and the approximate solutions of equation (3.33).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	1	$7.057224163 * 10^{-7}$	$7.057224163147 * 10^{-7}$
0.2	0.110517091807564	0.110516915298534	$1.765090298644 * 10^{-7}$
0.4	0.244280551632034	0.244280779871345	$2.282393114505 * 10^{-7}$
0.6000000000	0.404957642272801	0.404957642001080	$2.717201463831 * 10^{-10}$
0.8	0.596729879056508	0.596729668503415	$2.105530926277 * 10^{-7}$
1	0.824360635350064	0.824360625216032	$1.013403205796 * 10^{-8}$
1.2	1.093271280234305	1.093271491609722	$2.113754167254 * 10^{-7}$
1.4	1.409626895229333	1.409626915244021	$2.001468746293 * 10^{-8}$
1.5999999999	1.780432742793974	1.780432506067382	$2.367265912184 * 10^{-7}$
1.7999999999	2.213642800041254	2.21364297056188`	$1.705206256197 * 10^{-7}$
1.9999999999	2.718281828459044	2.718281085732449	$7.427265953374 * 10^{-7}$

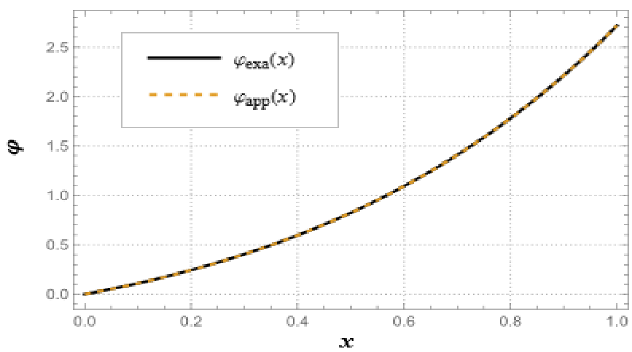

Figure 3.10(a): Exact and the approximate solutions of equation (3.27)

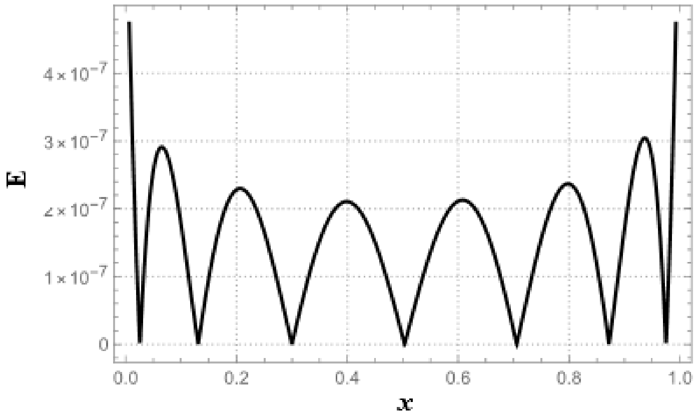

Figure 3.10(b): Absolute error for $n = 6$ of equation (3.27)

Example 3.11. we consider the following linear mixed Volterra-Fredholm integral equation of the second kind:

$$\varphi(x) = \sin(x) - \frac{1}{2}(x^2 + 2x) + \int_0^x \int_0^{\frac{\pi}{2}} (1 + yt)\varphi(t)dtdy \qquad (3.34)$$

We have the exact solution(Hasan & Suleiman, 2018) as

$$\varphi(x) = \sin(x)$$

We apply the algorithm of the proposed method with degree $n = 8$ for Bernstein polynomials, we get the approximate solution of equation (3.28) as,

$$\varphi_{app}(x) = 9.630 \times 10^{-9} + 0.999x + 0.005x^2 - 0.166x^3 + 0.007x^4 + 0.008x^5 + 0.001x^6 - 0.002x^7 + 0.001x^8$$

Table 3.11 and Figure 3.11(a) present some values of the exact and the approximate solutions calculated at some various values of x on $[0,1]$. The Bernstein polynomials method is applied for $n = 8$. The absolute error is calculated and has an order of approximately 10^{-9}. Hasan & Suleiman, (Hasan & Suleiman, 2018) solved equation (3.34) by technique is based on a polynomials with $n = 8$, where the absolute error was of order of approximately 10^{-7} . The comparison of absolute error of the proposed method and those in (Hasan & Suleiman, 2018), showing that the Bernstein polynomials method perform better than in (Hasan & Suleiman, 2018) with the same value of n.

Table 3.11. The approximate and the exact solutions of equation (3.34).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	0	$9.63059883254 * 10^{-9}$	$9.630598832543 * 10^{-9}$
0.2	0.099833416646828	0.099833413417822	$3.22900553517 * 10^{-9}$
0.4	0.198669330795061	0.198669332786160	$1.991099563003 * 10^{-9}$

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0.6000000000	0.295520206661339	0.295520208602582	$1.941243055192 * 10^{-9}$
0.8	0.389418342308650	0.389418341362777	$9.458731575762 * 10^{-10}$
1	0.479425538604203	0.479425536587015	$2.017187139546 * 10^{-9}$
1.2	0.564642473395035	0.564642472767768	$6.272666830398 * 10^{-10}$
1.4	0.644217687237691	0.644217688388225	$1.150534334470 * 10^{-9}$
1.5999999999	0.717356090899522	0.717356092331829	$1.432306606652 * 10^{-9}$
1.7999999999	0.783326909627483	0.783326909803904	$1.764212109733 * 10^{-10}$
1.9999999999	0.841470984807896	0.841470983687420	$1.120476267324 * 10^{-9}$

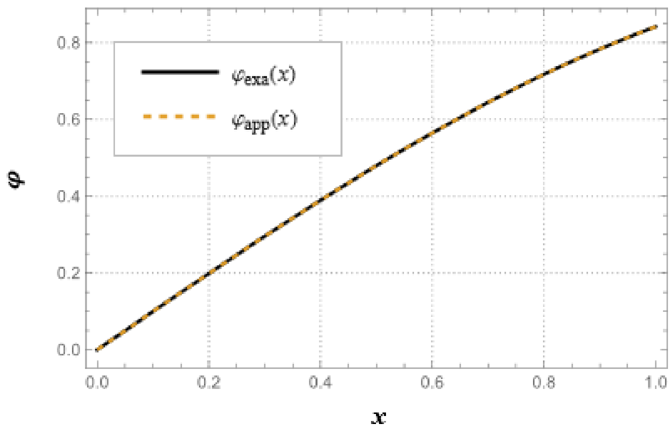

Figure 3.11(a): Exact and the approximate solutions of equation (3.34)

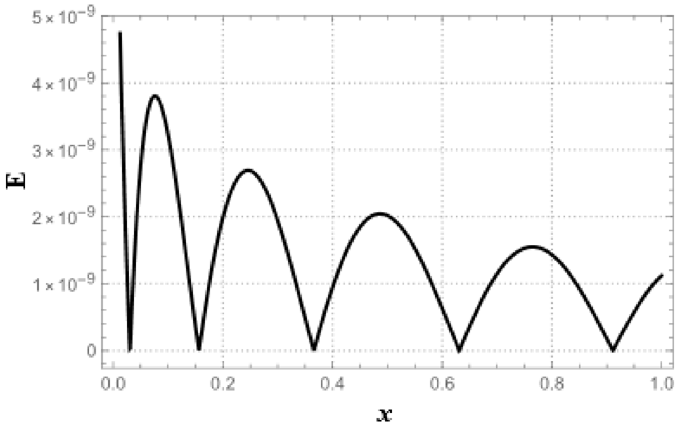

Figure 3.11(b): Absolute error for $n = 8$ of equation (3.34)

4 Numerical solution of linear Abel's integral equations by using Bernstein polynomials

4.1 Introduction

Abel's integral equation derived by the mathematician Niels Henrik Abel (1823). This equation is the oldest type of integral equation. Many branches of scientific fields depend on Abel's integral equation, such as X-ray radiography, atomic scattering microscopy, seismology, electron emission, radio astronomy, radar ranging, optical fibre evaluation, and plasma diagnostics (Huang et al., 2008) and (Vanani & Soleymani, 2013).

In this chapter, four kinds of linear Abel's integral equations are examined, such as generalized Abel integral equations, main generalized Abel integral equations and Abel's integral equations with both kinds. Additionally, we conduct an error analysis, evaluating eight numerical examples with absolute error metrics to compare the approximate with the exact ones. The results affirm the efficiency, applicability, and performance of the proposed numerical method.

4.2 Bernstein polynomials method for solving the generalized Abel's integral equation

We consider the generalized Abel's integral equation given by

$$f(x) = \lambda \int_0^x \frac{1}{(x-t)^\rho} \varphi(t)dt, \quad 0 < \rho < 1 \tag{4.1}$$

In this section, the generalized Abel's integral equation of the form (4.1) will be solved by using Bernstein polynomials. For this, we assume that the approximate solution $\varphi(x)$ of equation (4.1) is a function approximated by Bernstein polynomials as,

$$\varphi(x) \simeq \sum_{i=0}^{n} a_i \, B_{i,n}(x) \tag{4.2}$$

Where $B_{i,n}(x)$ are the Bernstein polynomials of degree n defined in equation (2.19) and a_i are coefficients to be determined. Substituting equation (4.2) into equation (4.1), we obtain

$$f(x) = \lambda \int_a^x \left[\frac{1}{(x-t)^\rho} \sum_{i=0}^{n} a_i B_{i,n}(t) \right] dt, x \in [a,b] and \; 0 < \rho < 1$$

$$f(x) = \sum_{i=0}^{n} a_i \left[\lambda \int_a^x \frac{1}{(x-t)^\rho} B_{i,n}(t)dt \right]$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1, \dots, n)$ and integrating both sides concerning x from a to b, we have

$$\int_a^b f(x)B_{j,n}(x)dx = \sum_{i=0}^{n} a_i \int_a^b \left[\lambda \int_a^x \frac{1}{(x-t)^{\rho}} B_{i,n}(t)dt\right] B_{j,n}(x)dx \tag{4.3}$$

Since above equation, there are two integrals, the inner integrand of the right side is a function of x and t and is integrated concerning to t from a to x. As a result, the outer integrand becomes a function of x only and integration concerning to x yields a constant. We suppose that:

$$Q_{ij} = \int_a^b \left[\lambda \int_a^x \frac{1}{(x-t)^{\rho}} B_{i,n}(t)dt\right] B_{j,n}(x)dx\,,\ \ i,j = 0,1, \dots, n$$

And

$$f_j = \int_a^b f(x)B_{j,n}(x)dx$$

Then equations (4.3) can be rewritten as

$$\sum_{i=0}^{n} a_i Q_{ij} = f_j \tag{4.4}$$

Thus, for each $j = 0,1,2, \dots, n$ we have a linear equation with $n + 1$ unknown coefficients $a_i, i = 0,1,2, \dots, n$. Therefore, the linear system (4.4) can be solved by any standard method to produce a_i and substituting these values back in (4.2), we get the approximate solution $\varphi(x)$ of the generalized Abel's integral equation.

Now, we discuss and analyses the convergence of the approximate solution.

4.2.1 Convergence analysis of the numerical solution of the generalized Abel's integral equation.

We show that the approximation solution from the Bernstein polynomials method converges with the exact solution of the generalized Abel's integral.

Theorem 4.1. (Convergence Theorem) (Zarei & Noeiaghdam, 2018) Let $f(x)$ be a function defined on $[a, b]$. Let $k(x, t)$ be in $[a, b] \times [0, t_1]$. Let $\varphi_{exa}(x)$ be the exact solution of the generalized Abel's integral equation (4.1) and $\varphi_{app}(x)$ be the approximation solution from the Bernstein polynomials method, Then

$$\lim_{n\to\infty} \|\varphi_{exa}(x) - \varphi_{app}(x)\| = 0$$

Proof. We start the proof the convergence from definition of the norm as,

$$\|\varphi_{exa}(x) - \varphi_{app}(x)\| = \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}} |\varphi_{exa}(x) - \varphi_{app}(x)|$$

$$= \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}} \left| f(x) + \lambda \int_a^b k(x,t)\varphi_{exa}(t)dt - \left[f(x) + \lambda \int_a^b k(x,t)\varphi_{app}(t)dt \right] \right|$$

$$= \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}} \left| \lambda \int_a^b k(x,t)\varphi_{exa}(t)dt - \left[\lambda \int_a^b k(x,t)\varphi_{app}(t)dt \right] \right|$$

$$= \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}} \left| \lambda \int_a^b k(x,t)\left[\varphi_{exa}(t) - \varphi_{app}(t)\right]dt \right|$$

We use Cauchy -Schwarz inequality given by equation (1.15), to obtain

$$\leq |\lambda| \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}} \int_a^b |k(x,t)|\left|\varphi_{exa}(t) - \varphi_{app}(t)\right|dt$$

If we suppose $Y = \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}} |k(x,t)|$**, t**hen we get

$$\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\| \leq |\lambda| Y \max_{\substack{x\in[a,b]\\ t\in[0,t_1]}} \int_a^b |k(x,t)|\left|\varphi_{exa}(t) - \varphi_{app}(t)\right|dt$$

Since $a \leq x \leq b$.

$$\leq \lambda Y(b-a)\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\|_\infty$$

We suppose $\alpha = \lambda Y \beta$, where $\beta = (b-a)$.

$$(1-\alpha)\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\|_\infty \leq 0$$

Therefore if $0 \leq \alpha \leq 1$ and $n \to \infty$,we have

$$\lim_{n\to\infty}\left\|\varphi_{exa}(x) - \varphi_{app}(x)\right\| = 0.$$

Theorem 4.2. (Existence Theorem) (Alipour & Rostamy, 2011) Let $f(x)$ be a function in the space $C[0,1]$, and $\epsilon > 0$ is arbitrary constant, then there exists polynomials $P_n(x)$ such that

$$|f(x) - P_n(x)| < \epsilon \ , x \in [a,b]$$

Proof. Suppose that $\mathcal{H} = l^2[0,1]$ is a Hilbert space with the inner product that is defined by $\langle f,g\rangle = \int_0^1 f(x)\, g(x)dx$ and $\{B_{0,n}, B_{1,n}, \dots, B_{n,n}\} \subset \mathcal{H}$ be the set of Bernstein polynomials n-th degree. Let $S_n = span\{B_{0,n}, B_{1,n}, \dots, B_{n,n}\}$ and f be an arbitrary element in $\mathcal{H}$. Since S_n is a finite dimensional and closed subspace, then S_n is a complete subset of $\mathcal{H}$. So f has the unique best approximation out of S_n such as $s_0 \in S_n$.

$$\exists s_0! \in S_n; \forall s \in S_n \|f - s_0\| \leq \|f - s\|$$

Where $\|f\|_2 = \sqrt{\langle f,f\rangle}$, Then $\forall s \in S_n (f - s_0, s) = 0$, Therefore, there exist unique coefficients $a_i, i = 0,1,\dots,n$ such that

$$f(x) \approx s_0 = \sum_{i=0}^{n} a_i B_{i,n}(x) = a^T \Phi_n(x)$$

Where $a^T = [a_0, a_1, \dots, a_n]$ can be obtained by

$$\langle f - a^T \Phi_n(x), B_{i,n}(x) \rangle = 0, i = 0,1, \dots n$$

$$a^T \langle \Phi_n(x), \Phi_n(x) \rangle = \langle f(x), \Phi_n(x) \rangle$$

Where $\langle f(x), \Phi_n(x) \rangle = \int_0^1 f(x)\, \Phi_n(x) dx = \left[\langle f, B_{0,n} \rangle, \langle f, B_{1,n} \rangle, \dots, \langle f, B_{n,n} \rangle \right]$

We define $\Psi = \langle \Phi_n(x), \Phi_n(x) \rangle$ that is a $(n+1)(n+1)$ matrix and it is called dual matrix of $\Phi_n(x)$ and we can obtain

$$\Psi_{i+1,j+1} = \int_0^1 B_{i,n}(x) B_{j,n}(x) dx, i, j = 0,1, \dots n$$

Recalling the integral Bernstein polynomials given by equation (2.10) leads to

$$\Psi_{i+1,j+1} = \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}$$

Lemma 4.1. Let $f: [0,1] \to \mathcal{R}$ is $(n+1)$ times continuously differentiable, and $S_n = span\{B_{0,n}, B_{1,n}, \dots, B_{n,n}\}$. If $a^T \Phi_n(x)$ is the best approximation f out of S then

$$\| f - a^T \Phi_n(x) \| \le \frac{\max_{x \in [0,1]} \left| f^{(n+1)}(x) \right|}{(n+1)! \sqrt{2n+3}}$$

Proof. We know that Set $\{1, x, \dots, x^n\}$ is a basis for polynomials space of degree n, Therefore we define

$$P_1(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0)$$

From Taylor expansion we have

$$|f(x) - P_1(x)| = \left| f^{n+1}(\xi_x) \frac{x^{n+1}}{n+1} \right|$$

Where $\xi_x \in [0,1]$. Since $a^T \Phi_n(x)$ is the best approximation f out of S, $P_1(x) \in S_n$, we have

$$\| f - a^T \Phi_n(x) \|^2_{l^2[0,1]} \le \| f - P_1(x) \|^2_{l^2[0,1]} = \int_0^1 f^{n+1}(\xi_x) \frac{x^{n+1}}{n+1} dx$$

$$\le \frac{\left(\max_{x \in [0,1]} \left| f^{(n+1)}(x) \right| \right)^2}{(n+1)^2! \, (2n+3)}$$

Then by taking square roots, the proof is complete, the previous Lemma shows that the error vanishes as $n \to \infty$.

Theorem 4.3 (Ray & Sahu, 2018) Let $\varphi_{exa}(x)$ be the exact solution of the generalized Abel's integral equation (4.1) and $\varphi_{app}(x) = \sum_{i=0}^{n} a_i\, B_{i,n}(x), i = 0,1,\dots,n$ be the approximation solution of equation (4.1), where $B_{i,n}(x)$ is the Bernstein polynomials of degree n. Then $\|E_n\| \to 0$, if the above an approximation solution converges to the exact solution of the generalized Abel's integral, when $n \to \infty$.

Proof. Consider $\varphi_{exa}(x)$ be the exact solution of the generalized Abel's integral equation (4.1), and $\varphi_{app}(x)$ be the approximation solution of equation (4.1), we have

$$f(x) = \lambda \int_0^x \frac{1}{(x-t)^\rho} \varphi_{app}(t)\, dt, \quad 0 < \rho < 1 \tag{4.5}$$

Now define the error functions E_n by subtracting equation (4.5) from equation (4.1) as follows:

$$E_n = f(x) - \lambda \int_0^x \frac{1}{(x-t)^\rho} \left[\varphi_{exa}(t) - \varphi_{app}(t)\right] dt$$

We have

$$\|E_n\| = \left\| f(x) - \lambda \int_0^x \frac{1}{(x-t)^\rho} \left[\varphi_{exa}(t) - \varphi_{app}(t)\right] dt \right\|$$

We use Cauchy -Schwarz inequality given by equation (1.15) to obtain

$$\|E_n\| \le \|f(x)\| - \lambda \int_0^x \left\| \frac{1}{(x-t)^\rho} \right\| \left\| \varphi_{exa}(t) - \varphi_{app}(t) \right\| dt$$

Now, $\|f(x)\|$ and $\left\|\frac{1}{(x-t)^\rho}\right\|$ are bound.

And from Theorem (4.1) we have $\left\|\varphi_{exa}(t) - \varphi_{app}(t)\right\| \to 0$ as $n \to \infty$, Hence $\|E_n\| \to 0$ as $n \to \infty$.

4.2.2 Numerical Results and Discussion

In this section, various examples are presented to illustrate the effectiveness of the method described. We compare the exact solution $\varphi_{exa}(x)$ with the approximate solution $\varphi_{app}(x)$, and we calculate the absolute error E.

Example 4.1. we consider generalized Abel's integral equation:

$$\int_0^x \frac{1}{(x-t)^{1/3}} \varphi(t) dt = x^{5/3}, \qquad x \in [0,1] \tag{4.6}$$

We have the exact solution (Yaghobifar et al., 2011) is:

$$\varphi_{exa}(x) = \frac{10}{9} x.$$

To determine the approximate solution of equation (4.6), we have

$$\varphi_{app}(x) = \sum_{i=0}^{4} a_i\, B_{i,4}(x) \tag{4.7}$$

Substituting equation (4.7) into equation (4.6), we obtain

$$\int_0^x \frac{1}{(x-t)^{1/3}} \sum_{i=0}^{4} a_i\, B_{i,4}(t)dt = x^{5/3}$$

$$\sum_{i=0}^{4} a_i \left[\int_0^x \frac{1}{(x-t)^{1/3}} B_{i,4}(t)dt\right] = x^{5/3}$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,2,3,4)$ and integrating both sides with respect to x from 0 to 1, we have

$$\sum_{i=0}^{4} a_i \int_0^1 \left[\int_0^x \frac{1}{(x-t)^{1/3}} B_{i,4}(t)dt\right] B_{j,4}(x)dx = \int_0^1 x^{5/3} B_{j,4}(x)dx \tag{4.8}$$

Now, we suppose that:

$$Q_{ij} = \int_0^1 \left[\int_0^x \frac{1}{(x-t)^{1/3}} B_{i,4}(t)dt\right] B_{j,4}(x)dx\,, \quad i,j = 0,1,2,3,4$$

And

$$f_j = \int_0^1 x^{5/3} B_{j,4}(x)dx\,, \quad j = 0,1,2,3,4$$

Then equations (4.8) can be rewritten as

$$\sum_{i=0}^{n} a_i Q_{ij} = f_j \tag{4.9}$$

We can calculate integrals Q_{ij} and f_j, and substitute them into equation (4.9), to get a linear system of equations

$$\frac{2187}{44660}a_0 + \frac{6561}{290290}a_1 + \frac{59049}{6676670}a_2 + \frac{177147}{66766700}a_3 + \frac{531441}{1135033900}a_4 = \frac{729}{52360}$$

$$\frac{34263}{580580}a_0 + \frac{2187}{51359}a_1 + \frac{1633}{66766700}a_2 + \frac{5963949}{567516950}a_3 + \frac{177147}{66766700}a_4 = \frac{243}{6545}$$

$$\frac{383211}{6676670}a_0 + \frac{3578661}{66766700}a_1 + \frac{6561}{158746}a_2 + \frac{1633689}{66766700}a_3 + \frac{65049}{6676670}a_4 = \frac{81}{1190}$$

$$\frac{3490803}{66766700}a_0 + \frac{31680801}{567516950}a_1 + \frac{3578661}{66766700}a_2 + \frac{2187}{51359}a_3 + \frac{6561}{290290}a_4 = \frac{9}{85}$$

$$\frac{53393859}{1135033900}a_0 + \frac{3490803}{66766700}a_1 + \frac{383211}{6676670}a_2 + \frac{34263}{580}a_3 + \frac{2187}{44660}a_4 = \frac{3}{20}$$

Therefore, we solve the linear system of equation to get

$a_0 = -1.469 \times 10^{-15}$, $a_1 = 0.277$, $a_2 = 0.555$, $a_3 = 0.833$, $a_4 = 1.111$

Then, we substitute these values back into equation (4.7) to get the approximate solution of equation (4.6) for $n = 4$ as follows

$$\varphi_{app}(x) = -1.469 \times 10^{-15} + 1.111x - 1.421 \times 10^{-13}x^2 + 2.411 \times 10^{-13}x^3 - 1.306 \times 10^{-13}x^4$$

Numerical results are listed in Table 4.1 illustrate the convergence of the approximate solution $\varphi_{app}(x)$ to the exact solution $\varphi_{ext}(x)$ at different values of x over [0,1], The absolute error is also calculated for each values and has an order of approximately 10^{-16}all over [0,1]. Table 4.3 and Figure b (4.1) confirm and justify the ability and efficiency of the method.

Table 4.1. Numerical results and absolute errors of equation (4.6).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	0	$-1.469727921 * 10^{-15}$	$1.469727921785 * 10^{-15}$
0.1	0.111111111111111	0.111111111111111	$2.081668171172 * 10^{-16}$
0.2	0.222222222222222	0.222222222222222	$3.053113317719 * 10^{-16}$
0.3000000004	0.333333333333333	0.333333333333333	$1.665334536937 * 10^{-16}$
0.4	0.444444444444444	0.444444444444443	$6.10622663543 * 10^{-16}$
0.5	0.555555555555555	0.555555555555554	$6.661338147750 * 10^{-16}$
0.6	0.666666666666666	0.666666666666666	$2.220446049250 * 10^{-16}$
0.7	0.777777777777777	0.777777777777778	$3.330669073875 * 10^{-16}$
0.7999999999	0.888888888888888	0.888888888888889	$5.551115123125 * 10^{-16}$
0.8999999999	1	0.999999999999999	$7.771561172376 * 10^{-16}$
0.9999999999	1.111111111111111	1.111111111111106	$4.440892098500 * 10^{-16}$

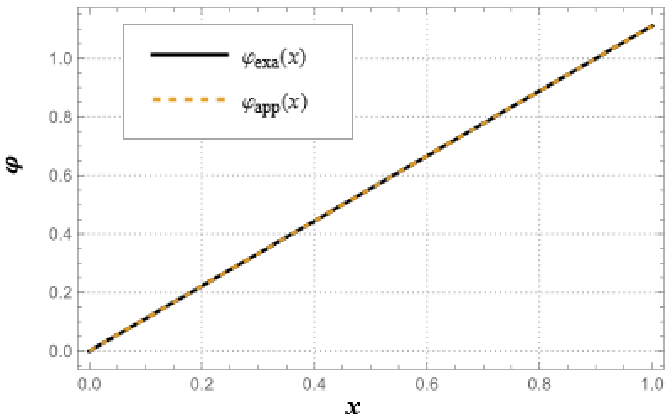

Figure 4.1(a): Exact and the approximate solutions of equation (4.6)

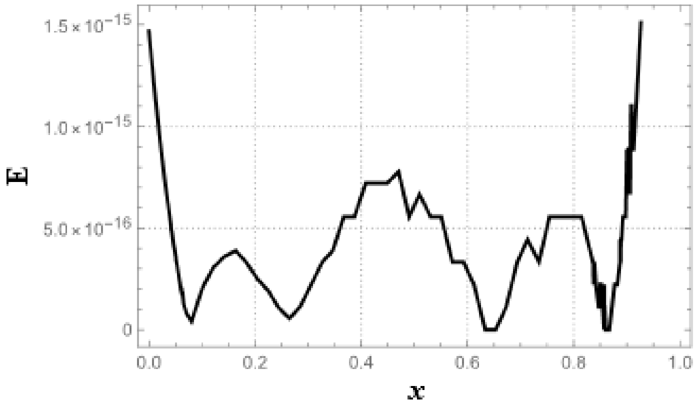

Figure 4.1(b): Absolute error for $n = 4$ of equation (4.6)

Example 4.2. Solve the following generalized Abel's integral equation:

$$\int_0^x \frac{1}{(x-t)^{1/4}}\varphi(t)dt = \frac{128}{231}x^{11/4}, \qquad x \in [0,1] \tag{4.10}$$

where the exact solution (Wazwaz, 2011a) obtained as,

$$\varphi_{ext}(x) = x^2$$

By approximating the unknown $\varphi(t)$ by the Bernstein polynomials of degree $n = 3$, the integral equation (4.10) is reduced to the following linear system,

$$\begin{bmatrix} \frac{2048}{35805} & \frac{8192}{322245} & \frac{65536}{7411635} & \frac{262144}{140821065} \\ \frac{24064}{322245} & \frac{2048}{39215} & \frac{1294336}{46940355} & \frac{65536}{7411635} \\ \frac{567488}{7411635} & \frac{3302144}{46940355} & \frac{2048}{39215} & \frac{8192}{322245} \\ \frac{10175696}{140821065} & \frac{567488}{7411635} & \frac{24064}{322245} & \frac{2048}{35805} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{65536}{13627845} \\ \frac{16384}{908523} \\ \frac{2048}{47817} \\ \frac{512}{6237} \end{bmatrix}$$

We solve the above system of equation, we get

$a_0 = -1.087 \times 10^{-15}$, $a_1 = 3.324 \times 10^{-15}$, $a_2 = 0.333$, $a_3 = 1$

Then, we substitute these values in the approximate solution when $n = 3$ as follows

$$\varphi_{app}(x) = -1.087 \times 10^{-15} + 1.323 \times 10^{-14}x + 0.999x^2 + 2.672 \times 10^{-14}x^3$$

Table 4.2 and Figure 4.2(a) present the exact and the approximate solutions calculated some various values of x over the interval [0,1]. The Bernstein polynomials method is applied for $n = 3$. The absolute error is calculated for some value and has an order of approximately 10^{-16}all over [0,1]. Figure 4.2(b) illustrates the behaviour of the absolute error for $n = 3$. Thus, one could claim that the generalized Abel's integral equation can be solved effectively by implementing the Bernstein polynomials.

Table 4.2. The approximate and the exact solutions of in equation (4.10).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	0	$-1.087498919 * 10^{-15}$	$1.087498919860 * 10^{-15}$
0.1	0.010000000000000	0.009999999999999	$1.023486850826 * 10^{-16}$
0.2	0.040000000000000	0.040000000000000	$3.191891195793 * 10^{-16}$
0.3000000004	0.090000000000000	0.090000000000000	$3.330669073876 * 10^{-16}$
0.4	0.160000000000000	0.160000000000000	$1.110223024625 * 10^{-16}$
0.5	0.25	0.249999999999999	$2.498001805406 * 10^{-16}$
0.6	0.36	0.359999999999999	$4.996003610813 * 10^{-16}$
0.7	0.489999999999999	0.489999999999999	$4.996003610813 * 10^{-16}$
0.7999999999	0.639999999999999	0.639999999999999	$1.110223024625 * 10^{-16}$
0.8999999999	0.809999999999999	0.810000000000000	$7.771561172376 * 10^{-16}$
0.9999999999	0.999999999999999	1.000000000000002	$2.442490654175 * 10^{-16}$

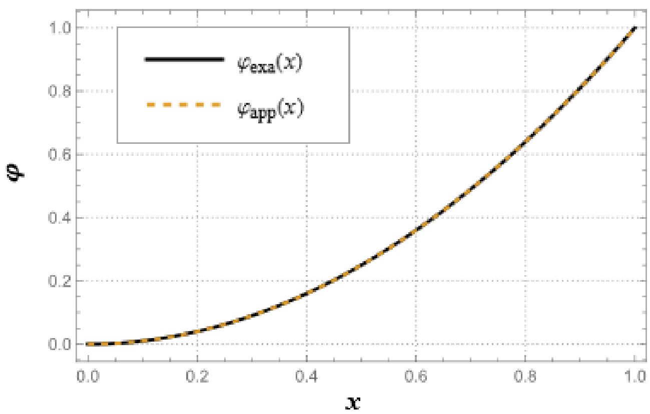

Figure 4.2(a): Exact and the approximate solutions of equation (4.10)

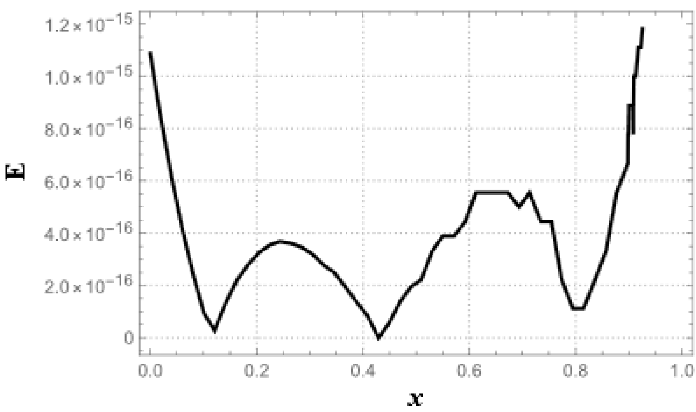

Figure 4.2(b): Absolute error for $n = 3$ of equation (4.10)

4.3 Bernstein polynomials method for solving the main generalized Abel equation

Definition 4.1. (Wazwaz, 2011)**:** The main generalized Abel's integral equations given by

$$f(x)=\int_0^x \frac{1}{|H(x)-H(t)|^{\rho}}\varphi(t)dt\,, \quad 0\;<\;\rho<1 \tag{4.11}$$

where $H(x)$ is strictly monotonically increasing and differentiable function in some interval $0<t<b$,and $h'(x)\neq 0$ for every t in the interval.

we apply the same steps of the Bernstein polynomials method for the main generalized Abel's integral equations.

we can use the following formula to obtain exact solution of the main generalized Abel's integral equations.

Lemma 4.1. (Wazwaz, 2011b): The analytical solution $\varphi(x)$ of equation (4.11) is given by formula as,

$$\varphi(x)=\frac{sin(\rho\pi)}{\pi}\frac{d}{dx}\int_0^x \frac{H'(t)f(t)}{|H(x)-H(t)|^{1-\rho}}dt\,, \qquad 0\;<\;\rho<1 \tag{4.12}$$

Proof. We multiply both sides of equation (4.11) by $\frac{1}{|H(x)-H(v)|^{1-\rho}}$,we have

$$\frac{f(x)}{|H(x)-H(v)|^{1-\rho}}=\frac{1}{|H(x)-H(v)|^{1-\rho}}\left[\int_0^x \frac{1}{|H(x)-H(t)|^{\rho}}\varphi(t)dt\right]$$

We integrate both sides with respect to x from 0 to w to get

$$\int_0^w \frac{f(x)}{|H(x)-H(v)|^{1-\rho}}dx \\ =\int_0^x \frac{\varphi(t)}{|H(x)-H(v)|^{1-\rho}}\left[\int_0^w \frac{1}{|H(x)-H(t)|^{\rho}}dx\right]dt \tag{4.13}$$

$$\int_0^w \frac{f(x)}{\big(H(x)-H(v)\big)^{1-\rho}}dx \\ =\int_0^x \varphi(t)\left[\int_0^w \frac{1}{\big(H(x)-H(v)\big)^{1-\rho}\big(H(x)-H(t)\big)^{\rho}}dx\right]dt \tag{4.14}$$

Let $\big(H(x)-H(t)\big)^{\rho}=(w-t)^{\rho}$, $\big(H(x)-H(v)\big)^{1-\rho}=(w-x)^{1-\rho}$,

$$I = \int_0^w \frac{dx}{(w-x)^{1-\rho}(w-t)^{\rho}} \tag{4.15}$$

We suppose $v = \frac{w-x}{w-t} \Rightarrow x = w - (w-t)v$ and $dx = -(w-t)dv$, then we can written equation (4.14) as follows

$$I = \int_0^1 \frac{(w-t)dv}{(w-t)(w-t)^{1-\rho}v^{1-\rho}(w-t)^{\rho}(1-v)^{\rho}}$$

$$= \int_0^1 \frac{dv}{v^{1-\rho}(1-v)^{\rho}}$$

$$= \int_0^1 v^{\rho-1}(1-v)^{-\rho}\, dv$$

$$= \beta(\rho, 1-\rho)$$

Where $\beta(\rho, 1-\rho)$ is the beta function.

$$= \frac{\Gamma(\rho)\Gamma(1-\rho)}{\Gamma(1)}$$

$$I = \frac{\pi}{\sin(\pi\rho)} \tag{4.16}$$

Now, we substitute equation (4.16) into equation (4.14) to get

$$\int_0^w \frac{f(x)}{\left(H(x)-H(v)\right)^{1-\rho}} dx = \int_0^w \varphi(t)\left[\frac{\pi}{\sin(\pi\rho)}\right] dt$$

$$\frac{\sin(\pi\rho)}{\pi}\int_0^w \frac{f(x)}{\left(H(x)-H(v)\right)^{1-\rho}} dx = \int_0^w \varphi(t)dt \tag{4.17}$$

We differentiate both sides of equation (4.17) to obtain

$$\varphi(x) = \frac{sin(\rho\pi)}{\pi}\frac{d}{dx}\int_0^x \frac{H(t)'f(t)}{|H(x)-H(t)|^{1-\rho}} dt$$

4.3.1 Numerical Results and Discussion

To verify the Bernstein polynomials method in solving linear generalized Abel's integral equation, we present two numerical examples featuring linear generalized Abel's integral equation with known exact solutions, the error estimation is provided to demonstrate the accuracy of the approximation.

Example 4.3. Solve the following the generalized Abel's integral equation:

$$\int_0^x \frac{1}{\sqrt{x^2-t^2}}\varphi(t)dt = \frac{2}{3}\pi x^3, \qquad x \in (0,2) \tag{4.18}$$

We obtain the exact solution by using the formula (4.11), SO

$\rho = \frac{1}{2}, f(x) = \frac{2}{3}\pi x^3$ and $H(x) = x^2$ and $H'(x) = 2x \neq 0$ for every x in $0 < x < 2$

Now, we use the formula given by equation (4.11), we obtain

$$\varphi_{ext}(x) = \frac{sin(\frac{\pi}{2})}{\pi}\frac{d}{dx}\int_0^x \frac{2t\frac{2}{3}\pi \mathrm{t}^3}{\sqrt{x^2-t^2}}dt$$

$$\varphi_{ext}(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{4}{3}\pi \mathrm{t}^4}{\sqrt{x^2-t^2}}dt$$

$$\varphi_{ext}(x) = \frac{1}{\pi}\frac{d}{dx}\left[\frac{\pi^2}{4}x^4\right] = \pi x^3$$

We apply the algorithm of the Bernstein polynomials method with degree $n = 5$ to obtain the approximate solution for $n = 5$ as

$$\varphi_{app}(x) = -5.834 \times 10^{-16} + 1.793 \times 10^{-14}x - 1.963 \times 10^{-13}x^2 + 3.142x^3$$
$$- 9.560 \times 10^{-13}x^4 + 4.448 \times 10^{-13}x^5$$

Table 4.3 displays the exact and the approximate solutions calculated at some values of x over $(0,2)$. The proposed method is applied with $n = 5$. The absolute error is also calculated at some value and has an order of approximately 10^{-14}. The results obtained in Table 4.5, Figure 4.3(a) and Figure 4.3(b) confirm that the Bernstein polynomials mothed approximates the solution in very good manner.

Table 4.3. Numerical results and absolute errors of equation (4.18).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0.2	0.025132741228718	0.025132741228717	$5.833642618435 * 10^{-16}$
0.4	0.201061929829746	0.201061929829746	$6.661338147750 * 10^{-16}$
0.6000000001	0.678584013175395	0.678584013175396	$1.942890293094 * 10^{-16}$
0.8	1.608495438637974	1.608495438637973	$5.551115123125 * 10^{-16}$
1	3.141592653589793	3.141592653589799	$1.332267629551 * 10^{-15}$
1.2	5.428672105403162	5.428672105403227	$5.773159728050 * 10^{-15}$
1.4	8.62053024145039	8.62053024145066`	$6.483702463810 * 10^{-14}$
1.5999999999	12.86796350910379	12.86796350910456	$2.700062395888 * 10^{-13}$
x	Exact solution	Approximate solution	Absolute error

	$\varphi_{exa}(x)$	$\varphi_{app}(x)$	$E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
1.7999999999	18.32176835573566	18.32176835573749	$7.762679388179 * 10^{-13}$
1.9999999999	25.13274122871833	25.13274122872209	$1.822542117224 * 10^{-12}$
0.2	0.025132741228718	0.025132741228717	$3.758771072179 * 10^{-12}$

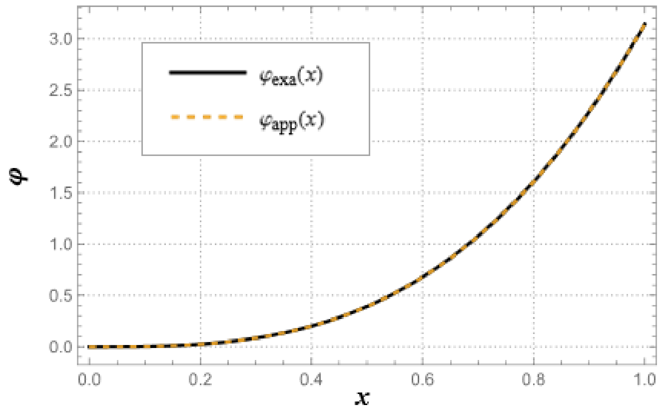

Figure 4.3(a): Exact and the approximate solutions of equation (4.18)

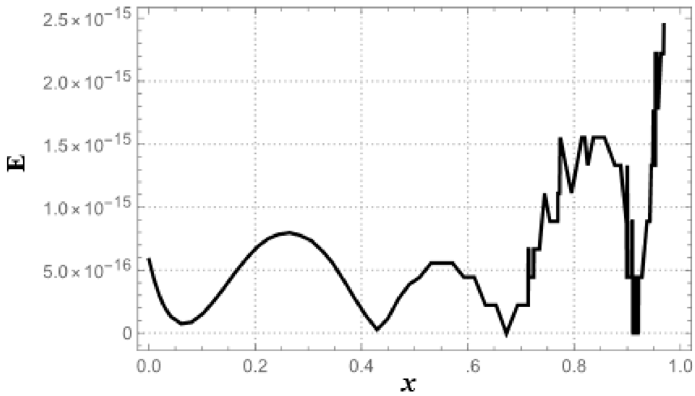

Figure 4.3(b): Absolute error for $n = 5$ of equation (4.18)

Example 4.4. Solve generalized Abel's integral equation:

$$\int_0^x \frac{1}{(x^5 - t^5)^{\frac{1}{6}}} \varphi(t)dt = \frac{6}{25} x^{25/6}, \qquad x \in (0,2) \tag{4.19}$$

We can get the exact solution of this integral equation from the formula given by equation (4.11) as

$$\varphi_{ext}(x) = x^4$$

We apply the algorithm of the Bernstein polynomials method with degree $n = 6$, we get the approximate solution of equation (4.19).

$$\varphi_{app}(x) = 3.508 \times 10^{-15} - 8.549 \times 10^{-14} x + 7.175 \times 10^{-13} x^2 - 1.632 \times 10^{-12} x^3 + x^4 - 7.9186 \times 10^{-13} x^5 + 1.425 \times 10^{-13} x^6$$

Numerical results are listed in Table 4.4 illustrate the convergence of the approximate solution $\varphi_{app}(x)$ to the exact solution $\varphi_{ext}(x)$ at different values of x over (0,2). The variations of the error of the approximate solution E are depicted in Figure 4.4(b). It is observed that the error of the approximate solution decreases as N increases. Table 4.4 and Figer 4.4(a) confirm that there are good results for this method. Hence it justifies the ability and efficiency of the Bernstein polynomials method.

Table 4.4. The approximate and the exact solutions of in equation (4.19).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	0	$3.5088709018 * 10^{-15}$	$3.508870901857 * 10^{-15}$
0.2	0.001600000000000	0.001600000000004	$4.467780312378 * 10^{-15}$
0.4	0.025600000000000	0.025600000000014	$1.467229115981 * 10^{-14}$
0.6000000001	0.129600000000008	0.129600000000018	$1.842970220877 * 10^{-14}$
0.8	0.409600000000002	0.409600000000017	$1.731947918415 * 10^{-14}$
1	1	1.000000000000016	$1.620925615952 * 10^{-14}$
1.2	2.0736	2.073600000000015	$1.554312234475 * 10^{-14}$
1.4	3.841599999999999	3.84160000000001`	$1.110223024625 * 10^{-14}$
1.5999999999	6.553599999999998	6.553599999999998	$8.881784197001 * 10^{-16}$
1.7999999999	10.49759999999999	10.49759999999998	$1.065814103640 * 10^{-14}$
1.9999999999	15.99999999999999	16.00000000000001	$2.131628207280 * 10^{-14}$

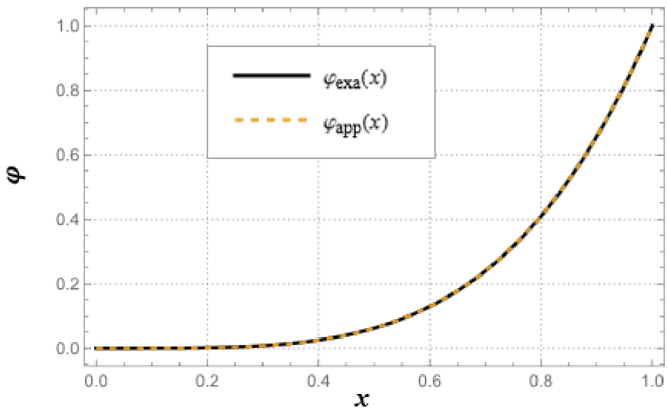

Figure 4.4(a): Exact and the approximate solutions of equation (4.19)

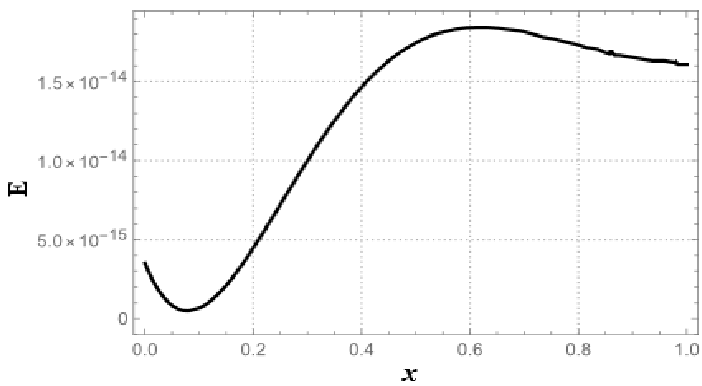

Figure 4.4(b): Absolute error for $n = 6$ of equation (4.19)

4.4 Bernstein polynomials method for solving Abel's integral equation

We consider Abel's integral equation of the first kind given as

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt,$$

and Abel's integral equation of the second kind as,

$$\varphi(x) = f(x) + \lambda \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt.$$

4.4.1 Numerical Results and Discussion

In this section, we present numerical examples to demonstrate the accuracy of the proposed Bernstein polynomials method and to verify the error estimates. The computations associated with the examples are performed by the Wolfram Mathematica version 14.1.

Example 4.5. Find the solution Abel's integral equation of the first kind:

$$\int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3), \qquad x \in [0,1] \qquad (4.20)$$

where the exact solution (Abdullah et al., 2024) is:

$$\varphi(x) = x^3 - x^2 + 1.$$

To determine the approximate solution of equation (4.20), we suppose

$$\varphi_{app}(x) = \sum_{i=0}^{4} a_i B_{i,4}(x) \qquad (4.21)$$

Substituting equation (4.21) into equation (4.20) leads to

$$\int_0^x \frac{1}{\sqrt{x-t}} \sum_{i=0}^{4} a_i B_{i,4}(t)dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3)$$

$$\sum_{i=0}^{4} a_i \left[\int_0^x \frac{1}{\sqrt{x-t}} B_{i,4}(t)dt\right] = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3)$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,\dots,4)$ and integrating both sides with respect to x from 0 to 1, we have,

$$\sum_{i=0}^{4} a_i \int_0^1 \left[\int_0^x \frac{1}{\sqrt{x-t}} B_{i,4}(t)dt\right] B_{j,4}(x)dx$$
$$= \int_0^1 \left[\frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3)\right] B_{j,4}(x)dx \quad (4.22)$$

Now, we suppose that:

$$Q_{ij} = \sum_{i=0}^{4} a_i \int_0^1 \left[\int_0^x \frac{1}{\sqrt{x-t}} B_{i,4}(t)dt\right] B_{j,4}(x)dx\,, \qquad i,j = 0,1,2,3,4$$

And

$$f_j = \int_0^1 \left[\frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3)\right] B_{j,4}(x)dx\,, \qquad j = 0,1,2,3,4$$

Then equations (4.22) can be rewritten as

$$\sum_{i=0}^{4} a_i Q_{ij} = f_j \quad (4.23)$$

We can calculate integrals Q_{ij} and f_j, and substitute them into equation (4.23), to get a system of linear equations

$$\frac{512}{5985}a_0 + \frac{4096}{101745}a_1 + \frac{8192}{508725}a_2 + \frac{32768}{6613425}a_3 + \frac{65536}{72747675}a_4 = \frac{236032}{1640925}$$
$$\frac{256}{2907}a_0 + \frac{2048}{2995}a_1 + \frac{274432}{6613425}a_2 + \frac{1359872}{72747675}a_3 + \frac{32768}{6613425}a_4 = \frac{7424}{35343}$$
$$\frac{4288}{56525}a_0 + \frac{172544}{2204475}a_1 + \frac{1024}{15675}a_2 + \frac{274432}{6613425}a_3 + \frac{8192}{508725}a_4 = \frac{326464}{1276275}$$
$$\frac{412768}{6613425}a_0 + \frac{5372}{72747675}a_1 + \frac{172544}{2204475}a_2 + \frac{2048}{29925}a_3 + \frac{4096}{101745}a_4 = \frac{374048}{1276275}$$
$$\frac{3764836}{72747675}a_0 + \frac{412768}{6613425}a_1 + \frac{4288}{56525}a_2 + \frac{256}{2907}a_3 + \frac{512}{5985}a_4 = \frac{96892}{294525}$$

Therefore, we solve the linear system of equation and get

$a_0 = 1$, $a_1 = 1$, $a_2 = 0.833$, $a_3 = 0.750$, and $a_4 = 1$.

Then, we substitute these values back into equation (4.21) to get the approximate solution of equation (4.20) when $n = 4$ as follows

$$\varphi_{app}(x) = 1 + 4.574 \times 10^{-14}x - x^2 + x^3 - 1.949 \times 10^{-13}x^4$$

Table 4.5 displays the exact and the approximate solutions calculated at some values of x over $[0,1]$. The proposed method is applied with $n = 4$, the absolute error is also calculated for each value and has an order of approximately 10^{-16}. Also, Figure 4.1(b) presents the behaviour of the absolute error for $n = 4$. The results obtained in Table 4.5 and Figure 4.5(a) confirm that the Bernstein polynomials mothed approximates the solution in the same way throughout the interval.

Table 4.5. Approximate solution and the exact solution of equation (4.20)

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	1	0.999999999999998	$1.887379141862 * 10^{-15}$
0.1	0.991	0.991000000000000	$7.771561172376 * 10^{-16}$
0.2	0.968	0.968000000000001	$1.221245327087 * 10^{-15}$
0.3000000004	0.936999999999999	0.937000000000000	$2.220446049253 * 10^{-16}$
0.4	0.904	0.903999999999999	$5.551115123125 * 10^{-16}$
0.5	0.875	0.874999999999999	$6.661338147759 * 10^{-16}$
0.6	0.856	0.856	0
0.7	0.853	0.853000000000000	$8.881784197001 * 10^{-16}$
0.7999999999	0.872	0.872000000000001	$1.332267629551 * 10^{-15}$
0.8999999999	0.918999999999999	0.918999999999999	0
0.9999999999	0.999999999999999	0.999999999999994	$5.218048215738 * 10^{-15}$

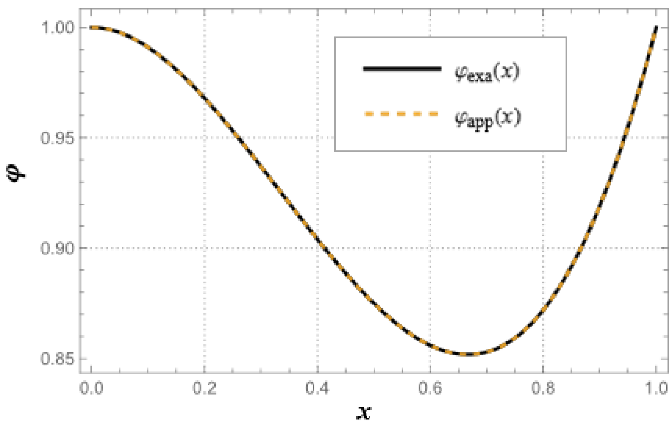

Figure 4.5(a): Exact and the approximate solutions of equation (4.20)

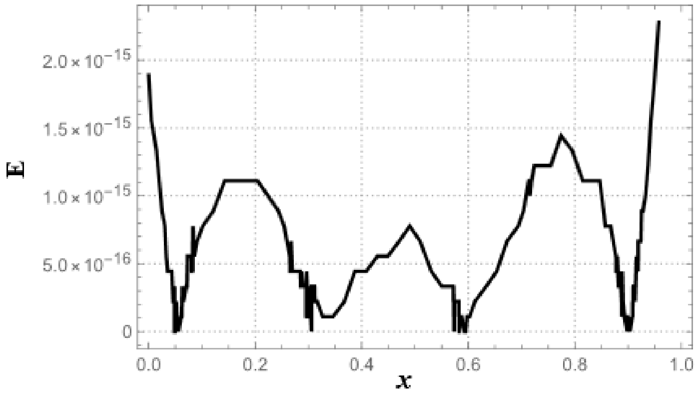

Figure 4.5(b): Absolute error for $n = 4$ of equation (4.20)

Example 4.6. Solve the following Abel's integral equation of the first kind:

$$\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt = 2\pi\sqrt{x}, \qquad x \in [0,1] \tag{4.24}$$

We can get the exact solution from the formula given by equation (4.11) as,

$$\varphi_{ext}(x) = \pi$$

We apply the algorithm of the Bernstein polynomials method with degree $n = 4$, we obtain the linear system

$$\frac{512}{5985}a_0 + \frac{4096}{101745}a_1 + \frac{8192}{508725}a_2 + \frac{32768}{6613425}a_3 + \frac{65536}{72747675}a_4 = 0.464$$

$$\frac{256}{2907}a_0 + \frac{2048}{29925}a_1 + \frac{274432}{6613425}a_2 + \frac{1359872}{72747675}a_3 + \frac{32768}{6613425}a_4 = 0.696$$

$$\frac{4288}{56525}a_0 + \frac{172544}{2204475}a_1 + \frac{1024}{15675}a_2 + \frac{274432}{6613425}a_3 + \frac{8192}{508725}a_4 = 0.87$$

$$\frac{4127688}{6613425}a_0 + \frac{5372672}{72747675}a_1 + \frac{172544}{2204475}a_2 + \frac{2048}{29925}a_3 + \frac{4096}{101745}a_4 = 1.015$$

$$\frac{3764836}{72747675}a_0 + \frac{412768}{6613425}a_1 + \frac{4288}{56525}a_2 + \frac{256}{2907}a_3 + \frac{512}{5985}a_4 = 1.142$$

We solve the linear system of equation, we get

$a_0 = 3.141$, $a_1 = 3.141$, $a_2 = 3.141$, $a_3 = 3.141$, and $a_4 = 3.141$

Then, the approximate solution when $n = 4$ as

$$\varphi_{app}(x) = 3.140 - 1.634 \times 10^{-13}x + 7.922 \times 10^{-13}x^2 - 1.329 \times 10^{-12}x^3 + 7.159 \times 10^{-13}x^4$$

Table 4.6 depicts the exact and the approximate solutions for equation (4.24) obtained by the Bernstein polynomials method. The absolute error is also calculated at some values and has an order of approximately 10^{-15} all over the interval [0,1]. Also, Figure 4.6

illustrates the convergence of the approximate solution to the exact solution for $n = 4$. The results confirm the effectiveness of the Bernstein polynomials method.

Table 4.6. Exact and the approximate solutions of in the equation (4.24).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	π	3.141592653589800	$7.549516567451 * 10^{-15}$
0.1	π	3.141592653589791	$2.220446049250 * 10^{-15}$
0.2	π	3.14159265358979	$3.108624468950 * 10^{-15}$
0.3000000004	π	3.141592653589793	0
0.4	π	3.141592653589795	$2.220446049250 * 10^{-15}$
0.5	π	3.141592653589795	$2.664535259103 * 10^{-15}$
0.6	π	3.141592653589793	$4.440892098506 * 10^{-16}$
0.7	π	3.14159265358979	$2.664535259103 * 10^{-15}$
0.7999999999	π	3.14159265358979	$3.108624468943 * 10^{-15}$
0.8999999999	π	3.141592653589796	$3.108624468954 * 10^{-15}$
0.9999999999	π	3.141592653589816	$2.353672812253 * 10^{-14}$

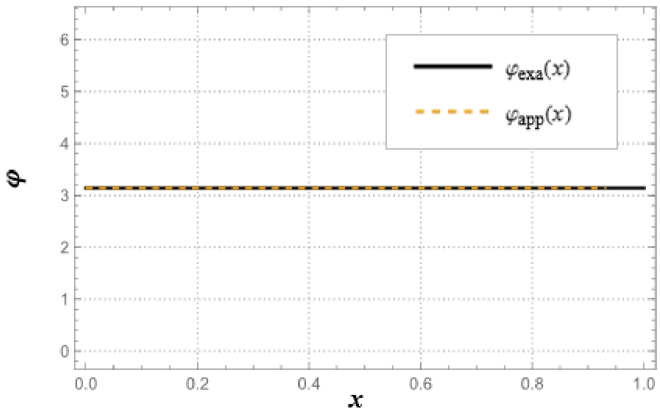

Figure 4.6(a): Exact and the approximate solutions of equation (4.24)

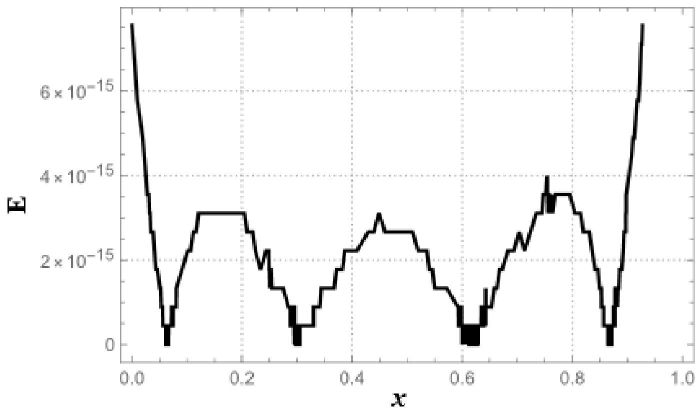

Figure 4.6(b): Absolute error for $n = 4$ of equation (4.24)

Example 4.7 Solve the following Abel's integral equation of the second kind:

$$\varphi(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt, \qquad x \in [0,1] \tag{4.25}$$

where the exact solution (Abdullah et al., 2024) as:

$$\varphi_{ext}(x) = x^2.$$

To determine the approximate solution of equation (4.25), we get

$$\varphi_{app}(x) = \sum_{i=0}^{4} a_i\, B_{i,4}(x) \tag{4.26}$$

Substituting equation (4.26) into equation (4.25), we obtain

$$\sum_{i=0}^{4} a_i\, B_{i,4}(x) + \int_0^x \frac{1}{\sqrt{x-t}} \sum_{i=0}^{4} a_i\, B_{i,4}(t)dt = x^2 + \frac{16}{15}x^{5/2}$$

$$\sum_{i=0}^{4} a_i \left[B_{i,4}(x) + \int_0^x \frac{1}{\sqrt{x-t}} \sum_{i=0}^{4} a_i\, B_{i,4}(t)dt\right] = x^2 + \frac{16}{15}x^{5/2}$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,\dots,4)$ and integrating both sides with respect to x from 0 to 1 leads to

$$\begin{aligned}\sum_{i=0}^{4} a_i \int_0^1 \left[B_{i,4}(x) + \int_0^x \frac{1}{\sqrt{x-t}} \sum_{i=0}^{4} a_i\, B_{i,4}(t)dt\right] B_{j,4}(x)dx \\ = \int_0^1 x^2 + \frac{16}{15}x^{5/2} B_{j,4}(x)dx\end{aligned} \tag{4.27}$$

Now, we suppose that:

$$Q_{ij} = \int_0^1 \left[B_{i,4}(x) + \int_0^x \frac{1}{\sqrt{x-t}} \sum_{i=0}^{4} a_i\, B_{i,4}(t)dt\right] B_{j,4}(x)dx, \quad i,j = 0,1,2,3,4$$

And

$$f_j = \int_0^1 x^2 + \frac{16}{15} x^{5/2} B_{j,4}(x)dx , \qquad j = 0,1,2,3,4$$

Then equations (4.27) can be rewritten as,

$$\sum_{i=0}^{4} a_i Q_{ij} = f_j \tag{4.28}$$

We can calculate integrals Q_{ij} and f_j, and substitute them into equation (4.28), to get a system of linear equations

$$\frac{1177}{5985} a_0 + \frac{6499}{67830} a_1 + \frac{40609}{1017450} a_2 + \frac{56837}{4408950} a_3 + \frac{362017}{145495350} a_4 = \frac{10531}{675675}$$
$$\frac{835}{5814} a_0 + \frac{188}{1425} a_1 + \frac{589357}{6613425} a_2 + \frac{1069144}{24249225} a_3 + \frac{56837}{4408950} a_4 = \frac{33641}{675675}$$
$$\frac{4829}{48450} a_0 + \frac{277519}{2204475} a_1 + \frac{13438}{109725} a_2 + \frac{589357}{6613425} a_3 + \frac{40609}{1017450} a_4 = \frac{7874}{75075}$$
$$\frac{930511}{13226850} a_0 + \frac{802248}{8083075} a_1 + \frac{277519}{2204475} a_2 + \frac{188}{1425} a_3 + \frac{6499}{67830} a_4 = \frac{3742}{20475}$$
$$\frac{7760617}{145495350} a_0 + \frac{930511}{13226850} a_1 + \frac{4829}{48450} a_2 + \frac{835}{5814} a_3 + \frac{1177}{5985} a_4 = \frac{449}{1575}$$

Therefore, we solve the above system of equation and get

$a_0 = -7.056 \times 10^{-17}$, $a_1 = 3.230 \times 10^{-15}$, $a_2 = 0.166$, $a_3 = 0.5$, $a_4 = 1$

Then, we substitute these values back into equation (4.26) we get the approximate solution of equation (4.25) for $n = 4$ as

$$\varphi_{app}(x) = -7.056 \times 10^{-17} + 1.320 \times 10^{-14} x + 0.999 x^2 + 8.038 \times 10^{-14} x^3 - 3.852 \times 10^{-14} x^4$$

Table 4.7 and Figure b (4.7) present the exact and the approximate solutions calculated at different nodes on the interval [0,1]. The proposed method is applied with $n = 4$. The absolute error is also calculated for each node and has an order of approximately 10^{-16}all over [0,1], Figure 4.7(b) illustrates the behaviour of the absolute error for $n = 4$. The results obtained in Table 4.7 and Figure 4.7(a) confirm that the Bernstein polynomials mothed approximates the solution in a very good manner.

Table 4.7. Exact and approximate solution of equation (4.25).

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0	0	$-7.056802041 * 10^{-17}$	$7.056802041611 * 10^{-17}$
0.1	0.010000000000000	0.010000000000000	$7.667477763817 * 10^{-16}$
0.2	0.040000000000000	0.040000000000000	$9.159339953157 * 10^{-16}$

x	Exact solution $\varphi_{exa}(x)$	Approximate solution $\varphi_{app}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$
0.3000000004	0.090000000000000	0.090000000000000	$7.216449660063 * 10^{-16}$
0.4	0.160000000000000	0.160000000000000	$4.440892098506 * 10^{-16}$
0.5	0.25	0.250000000000000	$2.220446049250 * 10^{-16}$
0.6	0.36	0.360000000000000	$1.110223024625 * 10^{-16}$
0.7	0.489999999999999	0.490000000000000	$1.110223024625 * 10^{-16}$
0.7999999999	0.639999999999999	0.640000000000000	$2.220446049250 * 10^{-16}$
0.8999999999	0.809999999999999	0.809999999999999	$1.110223024625 * 10^{-16}$
0.9999999999	0.999999999999999	0.999999999999998	$8.881784197001 * 10^{-16}$

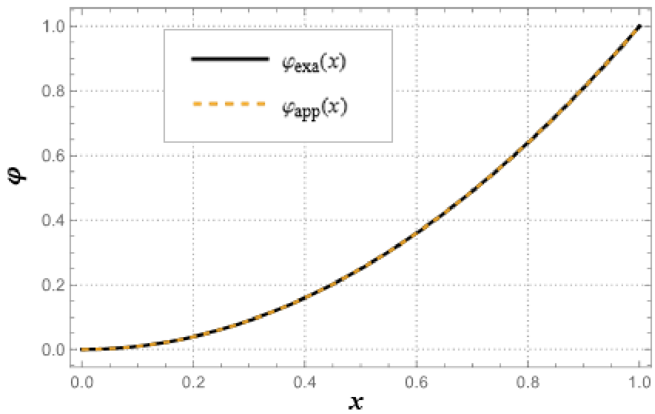

Figure 4.7(a): Exact and the approximate solutions of equation (4.25).

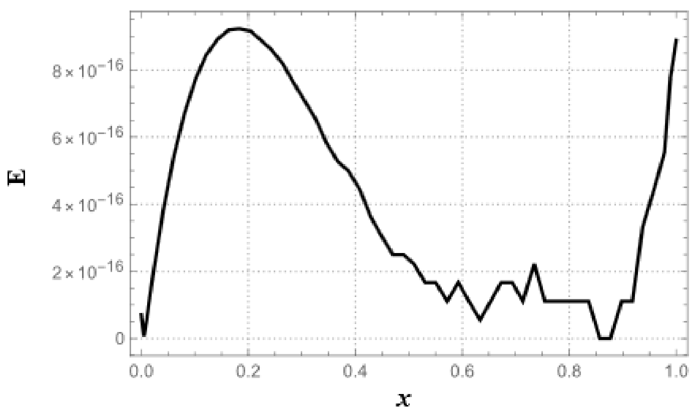

Figure 4.7(b): Absolute error for $n = 4$ of equation (4.25).

Example 4.8. Solve the second-kind Abel's integral equation:

$$4\varphi(x) = \frac{4}{\sqrt{x+1}} - \arcsin\left(\frac{1-x}{1+x}\right) + \frac{\pi}{2} - \int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt\,, \qquad (4.29)$$

$$0 \leq x < 1$$

where the exact solution(Sohrabi, 2011) is:

$$\varphi_{exa}(x) = \frac{1}{\sqrt{x+1}}$$

We apply the algorithm of the Bernstein polynomials method with degree $n = 7$, we obtain the approximate solution as follows

$$\varphi_{app}(x) = 0.999 - 0.499x + 0.374x^2 - 0.304x^3 + 0.236x^4 - 0.1503x^5 + 0.063x^6 - 0.012x^7$$

This problem has been solved by Bernstein polynomials mothed with different degree of Bernstein polynomials $n = 3,7,9$ which justifies the convergence of the numerical solution. The absolute error is also calculated for each node and has an order of approximately 10^{-9}all over $[0,1]$. It has been provided in Table 4.8, Figure 4.8(b), Figure 4.8(c) and Figure 4.8(d). Therefore, the obtained results justify the ability, efficiency and applicability of the method.

Table 4.8. Numerical results and absolute errors for equation (4.29).

x	Exact solution $\varphi_{exa}(x)$	Absolute error $E = \lvert\varphi_{exa}(x) - \varphi_{app}(x)\rvert$		
		$n = 3$	$n = 7$	$n = 9$
0	1	$8.39503699 * 10^{-4}$	$7.32639727 * 10^{-7}$	$2.04974774 * 10^{-8}$
0.1	0.95346258924	$2.25930580 * 10^{-4}$	$1.99656758 * 10^{-8}$	$6.22674535 * 10^{-9}$
0.2	0.91287092917	$2.95433676 * 10^{-4}$	$1.15798945 * 10^{-7}$	$5.16797948 * 10^{-9}$
0.300000004	0.87705801930	$2.98694508 * 10^{-5}$	$1.82191550 * 10^{-7}$	$2.86788648 * 10^{-9}$
0.4	0.84515425472	$2.07293803 * 10^{-4}$	$9.02981567 * 10^{-9}$	$1.37488409 * 10^{-9}$
0.5	0.81649658092	$2.59138193 * 10^{-4}$	$1.64126737 * 10^{-7}$	$4.38498937 * 10^{-9}$
0.6	0.79056941504	$1.16217476 * 10^{-4}$	$5.42918349 * 10^{-8}$	$3.15477088 * 10^{-9}$
0.7	0.76696498884	$1.23099952 * 10^{-4}$	$1.43050556 * 10^{-7}$	$7.7701356 * 10^{-10}$
0.799999999	0.74535599249	$2.79987317 * 10^{-4}$	$1.40457994 * 10^{-7}$	$3.78437325 * 10^{-9}$
0.899999999	0.72547625011	$1.14483890 * 10^{-4}$	$3.46121281 * 10^{-8}$	$4.91926743 * 10^{-9}$
0.999999999	0.70710678118	$6.60566458 * 10^{-4}$	$5.67770750 * 10^{-7}$	$1.74899380 * 10^{-8}$

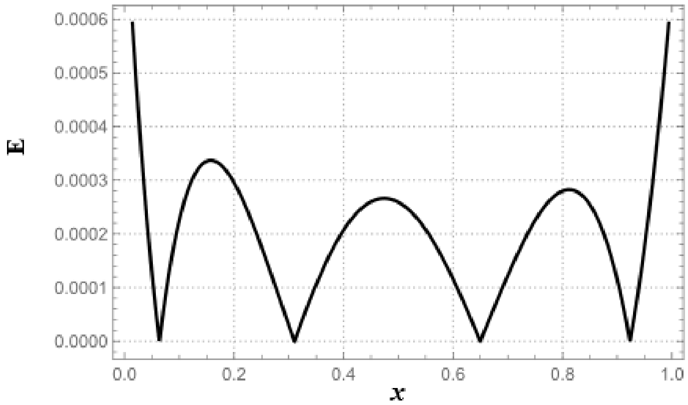

Figure 4.8(b): Absolute error for $n = 3$ of equation (4.29).

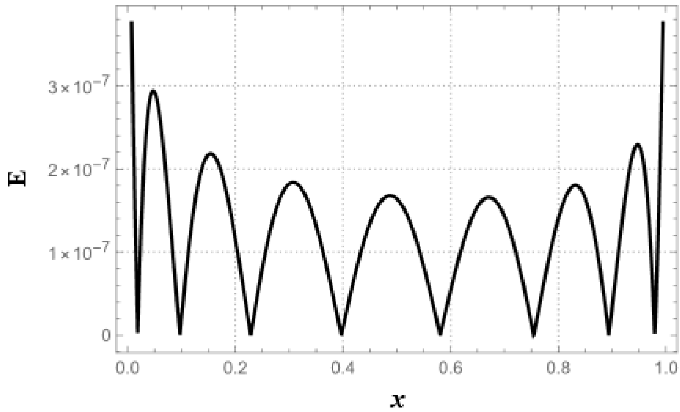

Figure 4.8(c): Absolute error for $n = 7$ of equation (4.29).

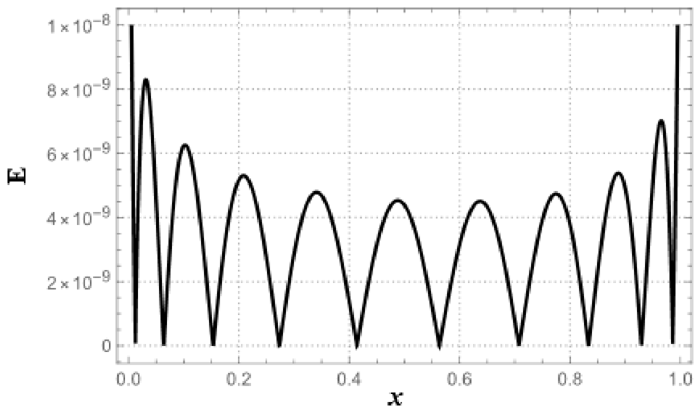

Figure 4.8(d): Absolute error for $n = 9$ of equation (4.29).

5 Conclusion and Future Work

5.1 Conclusion

In this dissertation, the unknown function in the IEs is approximated by an expansion in terms of Bernstein polynomials. Then by using the properties of Bernstein polynomials, the IEs are reduced to a system of linear algebraic equations, which can be easily solved to determine the coefficients of the Bernstein polynomials. These coefficients are then used to construct an approximate solution for the given integral equation. This method is characterized by its simplicity and effectiveness, especially for integral equations with complex kernels. To assess its efficiency, numerous examples were provided for different classes of linear IEs of both first and second kinds and with continuous and weakly singular kernels. Namely, we numerically solve IEs such as: Volterra IEs, Fredholm IEs, Volterra-Fredholm IEs, mixed Volterra-Fredholm IEs, Abel's IEs, generalized Abel's IEs, and main generalized Abel's IEs.

The third chapter, we have discussed the numerical results that are obtained from implementation of the solution algorithm which was programmed using the Wolfram Mathematica software: version 14.1 to solve IEs of the first and second such as Volterra IEs, Fredholm IEs, Volterra-Fredholm IEs, and mixed Volterra- Fredholm IEs, by the Bernstein polynomials method. The numerical results shown by numerous examples confirm the method is accurate and efficient. Example 3.1 showed that when increasing n, the approximate solution of the Volterra IEs converged to the exact solution. Also, Example 3.5 and Example 3.8 established the convergence of the numerical method.

In the fourth chapter, the Bernstein polynomials method was used to solve two examples of each of Abel IEs of the first and second kinds, generalized Abel IEs, and main generalized Abel IEs. The approximate solution was compared with the exact solution obtained from the formula (4.11), that is the numerical results showed that the approximate solution agrees with the exact solution and for small values of the Bernstein polynomials degree (n). The numerical results confirm and justify the ability and the reliability of the Bernstein polynomials method. Solutions obtained by the present method in compared to exact solutions admit remarkable efficiency and accuracy. The main advantages of this method are its efficiency and simple applicability.

5.2 Future Work

- ❖ The results of this dissertation establish the efficiency of the Bernstein method in solving different classes of integral equations. Future prospects include the extension of applying the Bernstein method for solving further types of IEs such as Volterra integro-differential IEs, Fredholm integro-differential IEs and Volterra - Fredholm integro-differential IEs, even more considering Integro-differential of fractional order.

- In this dissertation, we considered IEs of continuous or weakly-singular kernel. But in future work, we will consider other types of singularity such as IEs with strongly kernel or even with hypersingular kernels.
- Integrate the concepts of wavelets with Bernstein polynomials for function analysis and approximation.
- Adopt Bernstein Multiwavelets that expand upon wavelets to handle multi-dimensional data, providing a rigours framework for approximation and representation.

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Appendix A.

1. Mathematica algorithm for the examples in chapter 3

The code for solving the linear Volterra integral equations

Example 3.1.

```
n = 3;
a1 = 0;
b1 = 1;
c1 = x;
B[x_, i_] := Binomial[n, i] * ((x - a1) ^i * (b1 - x) ^ (n - i)) / (b1 - a1) ^n
k[t_, x_] := (Exp[x - t]);
a[x_] := 0;
f[x_] := Sin[x];
λ = 1;
K2[x_, j_] := Integrate[B[x, j] * f[x], {x, a1, b1}];
K3[j_, i_] := Integrate[(a[x] * B[x, i] + Integrate[λ * k[t, x] * B[t, i],
        {t, a1, c1}]) * B[x, j], {x, a1, b1}];
sol = Table[Total@Table[M[i] * K3[j, i], {i, 0, n}] == K2[x, j], {j, 0, n}];
Numerical[x_] := (func = Sum[M[i] * B[x, i], {i, 0, n}] /.
     NSolve[sol, Table[M[i], {i, 0, n}]] [[1]] // Simplify)
Exact[x_] := (Cos[x] - Sin[x]);
error[x_] := Abs[Exact[x] - Numerical[x]]
data = {};
For[i = 0, i ≤ 1, i = i + 0.1, row = {i};
 For[j = 1, j ≤ 3, j++, If[j == 1, AppendTo[row, Exact[i]]];
  If[j == 2, AppendTo[row, Numerical[i]]];
  If[j == 3, AppendTo[row, error[i]]]];
 AppendTo[data, row]]
Grid[data, Frame → All, Alignment → {Center, Center}]
Plot[error[x], {x, 0, 1}, PlotTheme → "Detailed", PlotStyle → Black]
Plot[{Exact[x], Numerical[x]}, {x, 0, 1}, PlotLegends → "Expressions",
 PlotTheme → "Detailed", PlotStyle → {Black, Dashed}]
```

The code for solving the linear Fredholm integral equations

Example 3.4.

```
n = 5;
a1 = 0;
b1 = 1;
B[x_, i_] := Binomial[n, i] * ((x - a1) ^i * (b1 - x) ^ (n - i)) / (b1 - a1) ^n
k[t_, x_] := ((x^2) * (t^2));
a[x_] := 1;
f[x_] := (0.9 * (x^2));
λ = (1 / 2);
K2[x_, j_] := Integrate[B[x, j] * f[x], {x, a1, b1}];
K3[j_, i_] := Integrate[(a[x] * B[x, i] - Integrate[λ * k[t, x] * B[t, i],
        {t, a1, b1}]) * B[x, j], {x, a1, b1}];
sol = Table[Total@Table[M[i] * K3[j, i], {i, 0, n}] == K2[x, j], {j, 0, n}];
Numerical[x_] := (func = Sum[M[i] * B[x, i], {i, 0, n}] /.
     NSolve[sol, Table[M[i], {i, 0, n}]] [[1]] // Simplify)
Exact[x_] := (x^2);
error[x_] := Abs[Exact[x] - Numerical[x]]
data = {};
For[i = 0, i ≤ 1, i = i + 0.1, row = {i};
 For[j = 1, j ≤ 3, j++, If[j == 1, AppendTo[row, Exact[i]]];
  If[j == 2, AppendTo[row, Numerical[i]]];
  If[j == 3, AppendTo[row, error[i]]]];
 AppendTo[data, row]]
Grid[data, Frame → All, Alignment → {Center, Center}]
Plot[error[x], {x, 0, 1}, PlotTheme → "Detailed", PlotStyle → Black, LabelStyle → Directive[Black, Bold]]
Plot[{Exact[x], Numerical[x]}, {x, 0, 1}, PlotLegends → "Expressions",
 PlotTheme → "Detailed", PlotStyle → {Black, Dashed}, LabelStyle → Directive[Black, Bold]]
```

The algorithm for solving the linear Volterra-Fredholm integral equations

Example 3.7.

```
n = 4;
a1 = 0;
b1 = 1;
c1 = x;
B[x_, i_] := Binomial[n, i] * ((x - a1) ^ i * (b1 - x) ^ (n - i)) / (b1 - a1) ^ n
k[t_, x_] := (x * (t + 1));
g[t_, x_] := ((2 * (x ^ 2) * t) + 1);
a[x_] := 1;
f[x_] := (((-2 / 5) * (x ^ 7)) - ((5 / 4) * (x ^ 4)) + (x ^ 3) - ((59 / 20) * (x)) + 1);
λ = 1;
h = 1;
K2[x_, j_] := Integrate[B[x, j] * f[x], {x, a1, b1}];
K3[j_, i_] := Integrate[(a[x] * B[x, i] - Integrate[λ * k[t, x] * B[t, i], {t, a1, b1
      Integrate[h * g[t, x] * B[t, i], {t, a1, c1}]) * B[x, j], {x, a1, b1}];
sol = Table[Total@Table[M[i] * K3[j, i], {i, 0, n}] == K2[x, j], {j, 0, n}];
φapp[x_] := (func = Sum[M[i] * B[x, i], {i, 0, n}] /.
     NSolve[sol, Table[M[i], {i, 0, n}]][[1]] // Simplify)
φexa[x_] := ((x ^ 3) + 1);
error[x_] := Abs[φexa[x] - φapp[x]]
data = {};
For[i = 0, i ≤ 1, i = i + 0.1, row = {i};
 For[j = 1, j ≤ 3, j++, If[j == 1, AppendTo[row, φexa[i]]];
  If[j == 2, AppendTo[row, φapp[i]]];
  If[j == 3, AppendTo[row, error[i]]]];
 AppendTo[data, row]]
Grid[data, Frame → All, Alignment → {Center, Center}]
Plot[error[x], {x, 0, 1}, PlotTheme → "Detailed", PlotStyle → Black]
Plot[{φexa[x], φapp[x]}, {x, 0, 1}, PlotLegends → "Expressions",
 PlotTheme → "Detailed", PlotStyle → {Black, Dashed}]
```

The algorithm for solving the linear mixed Volterra-Fredholm integral equations

Example 3.10.

```
n = 6;
a1 = 0;
b1 = 1;
c1 = x;
B[x_, i_] := Binomial[n, i] * ((x - a1) ^ i * (b1 - x) ^ (n - i)) / (b1 - a1) ^ n
k[t_, x_, r_] := (r);
a[x_] := 1;
f[x_] := (((x) * (Exp[x])) - ((1 / 2) * (x^2)));
λ = 1;
K2[x_, j_] := Integrate[B[x, j] * f[x], {x, a1, b1}];
K3[j_, i_] := Integrate[(a[x] * B[x, i] - Integrate[Integrate
        [λ * k[t, x, r] * B[t, i], {t, a1, b1}], {r, a1, c1}]) * B[x, j], {x, a1, b1}
sol = Table[Total@Table[M[i] * K3[j, i], {i, 0, n}] == K2[x, j], {j, 0, n}];
φapp[x_] := (func = Sum[M[i] * B[x, i], {i, 0, n}] /.
      NSolve[sol, Table[M[i], {i, 0, n}]][[1]] // Simplify)
φexa[x_] := ((x) * (Exp[x]));
error[x_] := Abs[φexa[x] - φapp[x]]
data = {};

 For[i = 0, i ≤ 1, i = i + 0.1, row = {i};
  For[j = 1, j ≤ 3, j++, If[j == 1, AppendTo[row, φexa[i]]];
   If[j == 2, AppendTo[row, φapp[i]]];
   If[j == 3, AppendTo[row, error[i]]]];
  AppendTo[data, row]]
 Grid[data, Frame → All, Alignment → {Center, Center}]
 Plot[error[x], {x, 0, 1}, PlotTheme → "Detailed", PlotStyle → Black]
 Plot[{φexa[x], φapp[x]}, {x, 0, 1}, PlotLegends → "Expressions",
  PlotTheme → "Detailed", PlotStyle → {Black, Dashed}]
```

2. Mathematica algorithm for the examples in chapter 4

The code for solving the generalized Abel's integral equations

Example 4.1.

```
n = 4;
a1 = 0;
b1 = 1;
c1 = x;
B[x_, i_] := Binomial[n, i] * ((x - a1) ^i * (b1 - x) ^ (n - i)) / (b1 - a1) ^n
k[t_, x_] := (1 / ((x - t) ^ (1 / 3)));
a[x_] := 0;
f[x_] := ((x) ^ (5 / 3));
λ = 1;
K2[x_, j_] := Integrate[B[x, j] * f[x], {x, a1, b1}];
K3[j_, i_] := Integrate[(a[x] * B[x, i] + Integrate[λ * k[t, x] * B[t, i],
        {t, a1, c1}]) * B[x, j], {x, a1, b1}];
sol = Table[Total@Table[M[i] * K3[j, i], {i, 0, n}] == K2[x, j], {j, 0, n}];
φapp[x_] := (func = Sum[M[i] * B[x, i], {i, 0, n}] /.
      NSolve[sol, Table[M[i], {i, 0, n}]] [[1]] // Simplify)
φexa[x_] := ((10 / 9) * x);
error[x_] := Abs[φexa[x] - φapp[x]]
data = {};|For[i = 0, i ≤ 1, i = i + 0.1, row = {i};
 For[j = 1, j ≤ 3, j++, If[j == 1, AppendTo[row, φexa[i]]];
  If[j == 2, AppendTo[row, φapp[i]]];
  If[j == 3, AppendTo[row, error[i]]]];
 AppendTo[data, row]]
Grid[data, Frame → All, Alignment → {Center, Center}]
Plot[error[x], {x, 0, 1}, PlotTheme → "Detailed", PlotStyle → Black]
Plot[{φexa[x], φapp[x]}, {x, 0, 1}, PlotLegends → "Expressions",
 PlotTheme → "Detailed", PlotStyle → {Black, Dashed}]
```

The algorithm for solving the main generalized Abel's integral equations

Example 4.3.

```
n = 5;
a1 = 0;
b1 = 1;
c1 = x;
B[x_, i_] := Binomial[n, i] * ((x - a1) ^i * (b1 - x) ^ (n - i)) / (b1 - a1) ^n
k[t_, x_] := (((x^2) - (t^2)) ^ (-1 / 2));
a[x_] := 0;
f[x_] := ((2 / 3) * Pi * (x^3));
λ = 1;
K2[x_, j_] := Integrate[B[x, j] * f[x], {x, a1, b1}];
K3[j_, i_] := Integrate[(a[x] * B[x, i] + Integrate[λ * k[t, x]
        * B[t, i], {t, a1, c1}]) * B[x, j], {x, a1, b1}];
sol = Table[Total@Table[M[i] * K3[j, i], {i, 0, n}] == K2[x, j], {j, 0, n}];
φapp[x_] := (func = Sum[M[i] * B[x, i], {i, 0, n}] /.
      NSolve[sol, Table[M[i], {i, 0, n}]] [[1]] // Simplify)
φexa[x_] := (Pi * (x^3));
error[x_] := Abs[φexa[x] - φapp[x]]
data = {}; For[i = 0, i ≤ 1, i = i + 0.1, row = {i};
 For[j = 1, j ≤ 3, j++, If[j == 1, AppendTo[row, φexa[i]]];
  If[j == 2, AppendTo[row, φapp[i]]];
  If[j == 3, AppendTo[row, error[i]]]];
 AppendTo[data, row]]
Grid[data, Frame → All, Alignment → {Center, Center}]
Plot[error[x], {x, 0, 1}, PlotTheme → "Detailed", PlotStyle → Black]
Plot[{φexa[x], φapp[x]}, {x, 0, 1}, PlotLegends → "Expressions",
 PlotTheme → "Detailed", PlotStyle → {Black, Dashed}]
```

The algorithm for solving Abel's integral equations

Example 4.8.

```
n = 3;
a1 = 0;
b1 = 1;
c1 = x;
B[x_, i_] := Binomial[n, i] * ((x - a1) ^i * (b1 - x) ^ (n - i)) / (b1 - a1) ^n
k[t_, x_] := (1 / ((x - t) ^ (1 / 2)));
a[x_] := 4;
f[x_] := (4 / ((x + 1) ^ (1 / 2)) - (ArcSin[(1 - x) / (1 + x)]) + (Pi / 2));
λ = 1;
K2[x_, j_] := Integrate[B[x, j] * f[x], {x, a1, b1}];
K3[j_, i_] := Integrate[(a[x] * B[x, i] + Integrate[λ * k[t, x]
        * B[t, i], {t, a1, c1}]) * B[x, j], {x, a1, b1}];
sol = Table[Total@Table[M[i] * K3[j, i], {i, 0, n}] == K2[x, j], {j, 0, n}];
φapp[x_] := (func = Sum[M[i] * B[x, i], {i, 0, n}] /.
     NSolve[sol, Table[M[i], {i, 0, n}]][[1]] // Simplify)
φexa[x_] := (1 / ((x + 1) ^ (1 / 2)));
error[x_] := Abs[φexa[x] - φapp[x]]
data = {}; For[i = 0, i ≤ 1, i = i + 0.1, row = {i};
 For[j = 1, j ≤ 3, j++, If[j == 1, AppendTo[row, φexa[i]]];
  If[j == 2, AppendTo[row, φapp[i]]];
  If[j == 3, AppendTo[row, error[i]]]];
 AppendTo[data, row]]
Grid[data, Frame → All, Alignment → {Center, Center}]
Plot[error[x], {x, 0, 1}, PlotTheme → "Detailed", PlotStyle → Black]
Plot[{φexa[x], φapp[x]}, {x, 0, 1}, PlotLegends → "Expressions",
 PlotTheme → "Detailed", PlotStyle → {Black, Dashed}]
```

الملخص

أغلب المسائل الفيزيائية تتم نمذجتها على شكل معادلات تفاضلية سواء كانت خطية أو غير خطية، ولكن لصعوبة حلها في كثير من الأحيان يتم اختزالها الى معادلات تكاملية مكافئة لتتم عملية ايجاد حلها بطرق أكثر فاعلية وسلاسة في التطبيق. تهدف هذه الرسالة الى الحصول على الحلول التقريبية لأصناف مختلفة من المعادلات التكاملية الخطية الآتية بنوعيها الأول والثاني: معادلات فولتيرا، معادلات فردهولم، معادلات فولتيرا- فردهولم التكاملية، معادلات فولتيرا-فردهولم المختلطة، معادلات أبل التكاملية الخطية من النوع الأول والثاني، معادلات أبل التكاملية المعممة ومعادلات آبل التكاملية المعممة الرئيسية. في هذا البحث تم استخدام كثيرات الحدود برنشتاين والتي تعتمد على تقريب الدالة المجهولة كتركيبة خطية من كثيرات حدود برنشتاين. وباستخدام خصائص هذه الكثيرات يتم اختزال المعادلة التكاملية إلى نظام من المعادلات الجبرية، والذي يمكن حلها بسهولة لإيجاد معاملات كثيرات حدود برنشتاين. هذه المعاملات تُستخدم بعد ذلك لبناء الحل التقريبي للمعادلة التكاملية المراد حلها. تتميز هذه الطريقة ببساطتها وفعاليتها، خاصةً للمعادلات التكاملية ذات النوى المعقَّدة. ولاختبار فاعلية هذه الطريقة تم تقديم العديد من الأمثلة لجميع الأصناف المذكورة أعلاه حيث تمت مقارنة الحل التقريبي بالحل المضبوط وأظهرت المقارنة تقارب الحل التقريبي الى الحل المضبوط ولقيم صغيرة لدرجة متعددات حدود برنشتين n. وهذا يثبت الفعالية والكفاءة العاليتين لهذه الطريقة العددية لحل المعادلات التكاملية. النتائج العددية المتحصل عليها دعمت الجانب التحليلي والنظريات المتعلقة بوجود، تقارب ووحدانية الحل التقريبي.

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رسالة ماجستير بعنوان

الحلول التقريبية لأصناف مختلفة من المعادلات التكاملية باستخدام كثيرات حدود برنيشتاين–دراسة تحليلية.

قدمت هذه الرسالة كأحد متطلبات الحصول على درجة الإجازة العالية (الماجستير)

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